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4D, $\mathcal{N} = 1$ supersymmetry genomics (I)

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Abstract: Presented in this paper the nature of the supersymmetrical representation theory behind 4D, $\mathcal{N} = 1$ theories, as described by component fields, is investigated using the tools of Adinkras and Garden Algebras. A survey of familiar matter multiplets using these techniques reveals they are described by two fundamental valise Adinkras that are given the names of the cis-Valise (c-V) and the trans-Valise (t-V). A conjecture is made that all off-shell 4D, $\mathcal{N} = 1$ component descriptions of supermultiplets are associated with two integers (n_c, n_t) — the numbers of c-V and t-V Adinkras that occur in the representation.

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Contents

1 Introduction

One of the long-standing unsolved problems in discussions of supersymmetrical theories is the notorious "off-shell problem." It is meant by this term that for a given set of propagating fields, there is currently no generally known prescription for how to augment this set with an additional finite number of fields (called "auxiliary fields") such that the algebra (see the appendix for the conventions used in this work)

$$\{ \mathrm{Q}_a^{\mathrm{I}} \, , \, \mathrm{Q}_b^{\mathrm{J}} \} \;=\; i\, 2\, \delta^{\mathrm{I}\,\mathrm{J}} (\gamma^\mu)_{a\,b}\, \partial_\mu \quad , \tag{1.1}$$

is satisfied for 'supercharges' Q_a that act non-trivially on both the propagating and auxiliary fields. They should act in such a way so as to *not* impose any particular dynamical equations on the propagating fields nor the auxiliary ones. Multiplets of both propagating and auxiliary fields that satisfy (1.1) and the conditions in the previous sentence are called "off-shell representations." Though the corresponding problem without auxiliary fields has long been resolved (see for example [1]), finding all such sets of fields in the off-shell case has been an unsolved problem since the birth of supersymmetry.

There is a general belief that this is an 'impossible' problem to solve. A widely accepted no-go theorem [2] has been derived that would seem to preclude the existence of such off-shell representations for a large class of *interesting* theories such as the 4D, $\mathcal{N} = 2$ Hypermultiplet [3], 4D, $\mathcal{N} = 4$ SUSY YM theory [4] and all 10D supersymmetrical theories that emerge as the low-energy zero-slope limits of superstring theories [5].

Two approaches have arisen to surmount the "off-shell problem." One of these is known as the 'harmonic superspace' approach [6] and the other is referred to as 'projective superspace' approach [7]. At the time of their creation, each approach seemed distinct but with the common feature of providing an off-shell description of the Hypermultiplet at the expense of using an infinite number of auxiliary fields. In the language of harmonic superspace the Hypermultiplet is known as the 'q-hypermultiplet.' Correspondingly, in the language of projective superspace the Hypermultiplet is known as the 'polar-hypermultiplet.'

Differences in the two approaches do exist. One of the most pointed is that only within the projective superspace approach is it possible to *easily* define a 4D, $\mathcal{N} = 1$ superfield truncation. However, there has been presented a proof [8] that any action in the harmonic superspace approach maybe engineered to yield an equivalent projective superspace formulation.

Though both of these two powerful methods have a long list of accomplishments to recommend them, it has long been the opinion of one of the authors (SJG) that these cannot represent the 'final' answer to the off-shell problem. One indication of this is the fact that though these infinite auxiliary-field extended technologies work, they only do so in a limited domain of theories. To our knowledge, neither of the methods has allowed a significant breakthrough for either 4D, $\mathcal{N} = 4$ SUSY YM theory nor any of the 10D supersymmetrical theories mentioned above. A final answer must deal with these cases also. In our opinion, to reach such a goal requires new tools and a new perspective.

For some time now, we have been developing two interlocking approaches in the effort to make progress on this problem. The first of these approaches [9, 10] involves what we refer to as the $\mathcal{GR}(\mathrm{d}, \mathcal{N})$ Algebra (or 'Garden Algebra') approach. Garden Algebras are real versions of Clifford Algebras that seem to provide the basic building blocks of a rigorous theory of representations for space-time SUSY. Our second approach is based on a set of diagrams we have named 'Adinkras' [11]. An Adinkra is essentially a weight space diagram (as is known for compact Lie algebras) but with the added feature of including the orbits of the distinct generators as they act on the states of any SUSY representation. Adinkras provide convenient graphical representations of Garden Algebras. A growing body of literature on these topics is being developed by a collaboration of computer scientists, mathematicians and theoretical physicists (the DFGHILM collaboration).

Some of this work has already uncovered unexpected relations between a classification of SUSY reps and graph theory [11], Filtered Clifford Algebras [12], graphical topology [13], and self-dual error-correcting codes [14] on the mathematics side. Alternately there has been presented a new off-shell 4D, $\mathcal{N} = 2$ hypermultiplet (the 'hyperplet') [15], new models for supersymmetrical quantum systems [16], and the first prepotential description of models [17] with an arbitrary degree of $\mathcal{N}$-extended SUSY on the physics side.

One of the current activities of the DFGHILM collaboration is the construction of a classification of supersymmetry representations up to and including $\mathcal{N} = 32$ systems. In the effort there is (what apparently seems to be) an incredible profusion of representations. So much so that we have been struck by the analogy with the problem of classifying genomes in biological systems. Building on this analogy, we have chosen to include the word 'genomics' in the title of this paper.

Although the work of [10] described a method (reduction on a 0-brane) by which the Adinkra/Garden Algebra description (the 'genetic description) of a supersymmetric representation can be uncovered, there has not to this point been a detailed presentation applying this technique to well-known 4D, $\mathcal{N} = 1$ systems more generally. In the following, we will obtain the genetic description of;

(a.) the off-shell and on-shell chiral multiplet,

(b.) the off-shell tensor multiplet,

(c.) the on-shell double-tensor multiplet, and

(d.) the off-shell and on-shell vector multiplet.

In a separate, but companion work, the complex linear multiplet and some other topics will be treated.

The structure of this work is as follows.

In section 2, reviews are given of the 4D, $\mathcal{N} = 1$ chiral, tensor, double tensor, and vector supermultiplets. This is mostly done to establish our notational conventions. In section 3, new results are presented. We carry out the reduction on a 0-brane of the 4D, $\mathcal{N} = 1$ supermultiplets discussed in the previous section. This truncation leads to 1D, $\mathcal{N} = 4$ supersymmetrical shadows and allows us to present an explicit derivation of the Garden Algebra matrices associated with these distinct multiplets. Though in the work of [10] it was stated that this procedure always leads to the discovery of the Garden Algebras matrices associated with each supersymmetric representation, the current work marks the first time this has been explicitly demonstrated for these familiar 4D, $\mathcal{N} = 1$ supermultiplets beyond the chiral multiplet. The work in section 4 is devoted to studying properties of the matrices associated with each of the multiplets. It is shown that (as expected) all the off-shell theories belong to a universal class of algebras. . . the Garden Algebras. On the other hand, the matrices associated with the on-shell theories do *not* possess features that lead to a universal identification. Thus the mathematical basis for understanding these in the context of matrix representation requires much more study. However, it is shown that there is one sharp distinction that can be made between 'generic' on-shell theories and 'pathogenic' on-shell theories. A definition is given for when two sets of Garden Algebra matrices are members of an equivalence class. Traces of the Garden Algebra matrices that respect this definition of equivalence are defined. Evidence is shown to support the proposal that the superspin of the 4D multiplets are encoded in the Garden Algebra and the initial steps toward defining characters are taken. A quantity, denoted by χ_0, is proposed as an actual character for the representations. The fifth section explores the construction of the Adinkras associated with each multiplet. By comparing the case of the chiral multiplet with the vector multiplet, it is shown what property of the Adinkra can be associated with χ_0. We give our conclusions in section six. At the end there are three appendices describing conventions, aspects of the structure we call $\mathcal{GR}(\mathrm{d}_{\mathbf{L}}, \mathrm{d}_{\mathbf{R}}, \mathcal{N})$, and a primer of Adinkra manipulation.

In closing this section, let us make a clear statement as what is and what is *not* the goal of this work. It is *not* a goal here to *solve* the problem of off-shell formulations of supersymmetric field theories in higher dimensions. The goal of the present work is much more modest. By applying simple 4D → 1D reduction (called "reduction on the 0-brane") we want to study the explicit results for a number of familiar 4D $\mathcal{N} = 1$ multiplets when by reduction, they are injected into the 'sea' of 1D representations that was discovered in the work of [9].

As was pointed out in [11], the number of representations in 1D (for a fixed number of supercharges) is enormously larger than those that arise as reductions of representations from higher dimensions. This raises the question of what distinguishes the generic 1D representations from those that are connected to higher D ones? As there is no over-arching theoretical guide for answering this question, it is paramount to know what are the explicit 1D representations that result from reduction. In a sense it is necessary to do a 'genomic scan' (i.e. to find the associated Adinkras and 'root superfield representations' [10]) of the reduced representations in order to compare these with generic 1D representations. We have chosen for this arena of study the 4D, $\mathcal{N} = 1$ theories.

We should point out that this current paper fills a hole in this line of investigations. With the exception of the work of [10], the DFGHILM collaboration has *not* produced works looking at the actual injection of higher dimensional multiplets into 1D. The work of the collaboration has been largely been directed to developing a firm mathematical background and understanding of the 1D theory. As a consequence, a number of results of this paper have not been seen previously and this paper is complementary to the general line of DFGHILM works.

2 Review of some 4D, $\mathcal{N} = 1$ multiplets

In each of the following subsections, a supersymmetric multiplet is presented in terms of its field content and supersymmetry transformation laws. Presentations are given for three off-shell multiplets as well as three on-shell multiplets.

2.1 Review of the 4D, $\mathcal{N} = 1$ Chiral Multiplet

The 4D, $\mathcal{N} = 1$ chiral multiplet is very well known to consist of a scalar A, a pseudoscalar B, a Majorana fermion ψ_a, a scalar auxiliary field F, and a pseudoscalar auxiliary field G. A convenient way to express the supersymmetry variation of these component fields is by first regarding them as the lowest component of a superfield (denoted by the same symbol) and then expressing the action of the superspace covariant derivative D_a acting on each. As we have included the auxiliary fields F and G, necessarily it is the off-shell theory under consideration.

The supersymmetry variations can be cast into the form of a set of specifications of the superspace 'covariant derivative' acting on a set of superfields. We have in our conventions

$$
\begin{aligned}
\mathrm{D}_a A &= \psi_a \quad , \\
\mathrm{D}_a B &= i\,(\gamma^5)_a{}^b\,\psi_b \quad , \\
\mathrm{D}_a \psi_b &= i\,(\gamma^\mu)_{a\,b}\,\partial_\mu A \;-\; (\gamma^5\gamma^\mu)_{a\,b}\,\partial_\mu B \;-\; i\,C_{a\,b}\,F \;+\; (\gamma^5)_{a\,b} G \;, \\
\mathrm{D}_a F &= (\gamma^\mu)_a{}^b\,\partial_\mu\,\psi_b \quad , \\
\mathrm{D}_a G &= i\,(\gamma^5\gamma^\mu)_a{}^b\,\partial_\mu\,\psi_b \quad .
\end{aligned}
\tag{2.1}
$$

A direct calculation shows that

$$
\begin{aligned}
\{\, \mathrm{D}_a\,,\, \mathrm{D}_b \,\}\,A &= i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,A \quad , \quad \{\, \mathrm{D}_a\,,\, \mathrm{D}_b \,\}\,B \;=\; i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,B \quad , \\
\{\, \mathrm{D}_a\,,\, \mathrm{D}_b \,\}\,\psi_c &= i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,\psi_c \quad , \\
\{\, \mathrm{D}_a\,,\, \mathrm{D}_b \,\}\,F &= i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,F \quad , \quad \{\, \mathrm{D}_a\,,\, \mathrm{D}_b \,\}\,G \;=\; i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,G \quad .
\end{aligned}
\tag{2.2}
$$

As expected, the algebra of (1.1) is satisfied independently of the field upon which it is evaluated.

The simplest version of the on-shell theory occurs by simply setting $F = G = 0$ in (2.1) and (2.2). Thus (2.1) is replaced by

$$
\begin{aligned}
\mathrm{D}_a A &= \psi_a \quad , \\
\mathrm{D}_a B &= i\,(\gamma^5)_a{}^b\,\psi_b \quad , \\
\mathrm{D}_a \psi_b &= i\,(\gamma^\mu)_{a\,b}\,\partial_\mu A \;-\; (\gamma^5\gamma^\mu)_{a\,b}\,\partial_\mu B \quad .
\end{aligned}
\tag{2.3}
$$

Using (2.3), a direct calculation shows that

$$
\begin{aligned}
\{\, \mathrm{D}_a\,,\, \mathrm{D}_b \,\}\,A &= i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,A \quad , \quad \{\, \mathrm{D}_a\,,\, \mathrm{D}_b \,\}\,B \;=\; i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,B \quad , \\
\{\, \mathrm{D}_a\,,\, \mathrm{D}_b \,\}\,\psi_c &= i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,\psi_c \;-\; i\,(\gamma^\mu)_{a\,b}\,(\gamma_\mu\gamma^\nu)_c{}^d\partial_\nu\,\psi_d \quad .
\end{aligned}
\tag{2.4}
$$

The first two of these equations have the same form as (1.1) in the case where $\mathcal{N} = 1$. However, the third term immediately above can be expressed as

$$
\begin{aligned}
\{\, \mathrm{D}_a\,,\, \mathrm{D}_b \,\}\,\psi_c &= i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,\psi_c \;+\; i\,2\,(\gamma^\mu)_{a\,b}\,(\gamma_\mu)_c{}^d\mathcal{K}_d(\psi) \;, \\
\mathcal{K}_c(\psi) &= -\,\frac{1}{2}\,(\gamma^\nu)_c{}^d\partial_\nu\,\psi_d \quad ,
\end{aligned}
\tag{2.5}
$$

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where $\mathcal{K}_c$ measures the 'non-closure' of the algebra. It is also seen that the relations

$$\mathcal{K}_c(\psi) \;=\; -\,\frac{1}{2}\,\mathrm{D}_c F \quad , \quad \mathcal{K}_c(\psi) \;=\; i\,\frac{1}{2}\,(\gamma^5)_c{}^d\,\mathrm{D}_d G \tag{2.6}$$

are satisfied. This is important for the consistency of the truncation in (2.4) with regard to the starting point in (2.2). If we set $F = G = 0$ in (2.2) then it is consistent to set $\mathcal{K}_c = 0$ in (2.4)–(2.6).

This is the essence of the "Off-Shell Problem." Namely, if we begin only knowing (2.3), how would we systematically go about finding out that it is required to add F and G as in (2.1)? A related question is, "Is the addition of F and G unique?" (The answer to this second question is known to be, "No." This will be discussed in a companion work to accompany this paper.)

2.2 Review of the 4D, $\mathcal{N} = 1$ Tensor Multiplet

The 4D, $\mathcal{N} = 1$ tensor multiplet consists of a scalar φ, a second-rank skew symmetric tensor, $B_{\mu\nu}$, and a Majorana fermion χ_a. Their supersymmetry variations can be cast in the forms

$$\begin{aligned}
\mathrm{D}_a\varphi &= \chi_a \quad , \\
\mathrm{D}_a B_{\mu\nu} &= -\,\frac{1}{4}([\,\gamma_\mu\,,\,\gamma_\nu\,])_a{}^b\,\chi_b \quad , \\
\mathrm{D}_a\chi_b &= i\,(\gamma^\mu)_{a\,b}\,\partial_\mu\varphi \;-\; (\gamma^5\gamma^\mu)_{a\,b}\,\epsilon_\mu{}^{\rho\,\sigma\,\tau}\partial_\rho B_{\sigma\,\tau} \quad .
\end{aligned} \tag{2.7}$$

The commutator algebra for the D-operator calculated from (2.7) takes the form

$$\begin{aligned}
\{\,\mathrm{D}_a\,,\,\mathrm{D}_b\,\}\,\varphi &= i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,\varphi \ , \\
\{\,\mathrm{D}_a\,,\,\mathrm{D}_b\,\}\,B_{\mu\,\nu} &= i\,2\,(\gamma^\rho)_{a\,b}\,\partial_\rho\,B_{\mu\,\nu} \;+\; \partial_\mu\,q_{\nu\;a\,b} \;-\; \partial_\nu\,q_{\mu\;a\,b} \ , \\
\{\,\mathrm{D}_a\,,\,\mathrm{D}_b\,\}\,\chi_c &= i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,\chi_c \ , \quad q_{\mu\;a\,b} \;\equiv\; i\,2\,(\gamma^\nu)_{a\,b}\left[\,B_{\mu\,\nu} + \frac{1}{2}\eta_{\mu\,\nu}\,\varphi\,\right] \quad .
\end{aligned} \tag{2.8}$$

The second line in (2.8) is interesting. On a first glance, it appears that the two final q-dependent parts are 'non-closure' terms as seen in the on-shell chiral multiplet. Let us multiply the middle line of (2.8) by parameters ϵ_1^a and ϵ_2^b to obtain

$$\begin{aligned}
\epsilon_1^a\,\epsilon_2^b\,\{\,\mathrm{D}_a\,,\,\mathrm{D}_b\,\}\,B_{\mu\,\nu} &= i\,2\,\epsilon_1^a\,\epsilon_2^b\,(\gamma^\rho)_{a\,b}\,\partial_\rho\,B_{\mu\,\nu} \;+\; \partial_\mu\,\mathrm{v}_\nu \;-\; \partial_\nu\,\mathrm{v}_\mu \\
\text{where } \mathrm{v}_\mu &\equiv i\,2\,\epsilon_1^a\,\epsilon_2^b\,(\gamma^\nu)_{a\,b}\left[\,B_{\mu\,\nu} + \frac{1}{2}\eta_{\mu\,\nu}\,\varphi\,\right] \quad .
\end{aligned} \tag{2.9}$$

Since $B_{\mu\nu}$ is anti-symmetric, it is possible to define a 'gauge' variation denoted by $\delta_G^{(2)}(\ell)$ that acts upon it according to

$$\delta_G^{(2)}(\ell)\,B_{\mu\,\nu} \;=\; \partial_\mu\,\ell_\nu \;-\; \partial_\nu\,\ell_\mu \quad , \tag{2.10}$$

and if we identify $\ell_\mu = \mathrm{v}_\mu$, then (2.9) may be expressed as

$$\begin{aligned}
\epsilon_1^a\,\epsilon_2^b\,\{\,\mathrm{D}_a\,,\,\mathrm{D}_b\,\}\,B_{\mu\,\nu} &= i\,2\,\epsilon_1^a\,\epsilon_2^b\,(\gamma^\rho)_{a\,b}\,\partial_\rho\,B_{\mu\,\nu} \;+\; \delta_G^{(2)}(\mathrm{v})\,B_{\mu\,\nu} \\
&\equiv \xi^\rho\partial_\rho\,B_{\mu\,\nu} \;+\; \delta_G^{(2)}(\mathrm{v})\,B_{\mu\,\nu} \quad .
\end{aligned} \tag{2.11}$$

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These equations inform us that any theory possessing the symmetries described by (2.7) must also possess the symmetries described on the r.h.s. (right hand side) of (2.11). The first of these is simple translation symmetry. The second is identifiable as the gauge symmetry of an anti-symmetric rank two tensor field. Finally, we observe that there are no Lorentz-covariant truncations of the fields in the tensor multiplet. So it is not possible to define an 'on-shell' version of this multiplet as it was with the chiral multiplet.

2.3 4D, $\mathcal{N} = 1$ Double Tensor Multiplet

Though little known, the 4D, $\mathcal{N} = 1$ "double tensor" multiplet is quite old [18]. To motivate the consideration of this multiplet, it is useful to compare (2.3) with (2.7), looking for differences and similarities. Immediately, one glaring difference is that the pseudoscalar field B in the chiral multiplet is replaced by the 2-form $B_{\mu\nu}$ in the tensor multiplet. This obviously motivates the query, "What would occur if *both* A *and* B were replaced by 2-forms?" The 4D, $\mathcal{N} = 1$ double tensor multiplet consists of two second-rank skew symmetric tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$ along with a Majorana fermion Λ_a. Thus we arrive at the double tensor multiplet with supersymmetry variations taking the forms

$$\begin{aligned} \mathrm{D}_a X_{\mu\nu} &= i\,\frac{1}{4}(\gamma^5[\,\gamma_\mu\,,\,\gamma_\nu\,])_a{}^b\,\Lambda_b \quad , \\ \mathrm{D}_a Y_{\mu\nu} &= -\,\frac{1}{4}([\,\gamma_\mu\,,\,\gamma_\nu\,])_a{}^b\,\Lambda_b \quad , \\ \mathrm{D}_a \Lambda_b &= i\,(\gamma^\mu)_{a\,b}\,\epsilon_\mu{}^{\rho\,\sigma\,\tau}\partial_\rho X_{\sigma\,\tau} \;-\; (\gamma^5\gamma^\mu)_{a\,b}\,\epsilon_\mu{}^{\rho\,\sigma\,\tau}\partial_\rho Y_{\sigma\,\tau} \quad . \end{aligned} \tag{2.12}$$

Upon comparing (2.4) with (2.12), it is clear that the first two equations in the former will become the first two equations of the latter if we perform the replacements

$$A \;\to\; X_{\mu\nu} \;,\quad B \;\to\; Y_{\mu\nu} \;,\quad \psi_a \;\to\; i\,\frac{1}{4}(\gamma^5[\,\gamma_\mu\,,\,\gamma_\nu\,])_a{}^b\,\Lambda_b \quad . \tag{2.13}$$

Curiously though, if the replacements in (2.13) are inserted into the final line in (2.4), we obtain

$$\begin{aligned} \mathrm{D}_a \Lambda_b \;=\; & i\,\frac{1}{6}\,(\gamma^\mu)_{a\,b}\,[\,\epsilon_\mu{}^{\rho\,\sigma\,\tau}\,\partial_\rho X_{\sigma\,\tau} \;-\; 2\,\partial^\nu Y_{\mu\nu}\,] \\ & +\;\frac{1}{6}\,(\gamma^5\gamma^\mu)_{a\,b}\,[\,\epsilon_\mu{}^{\rho\,\sigma\,\tau}\,\partial_\rho Y_{\sigma\,\tau} \;+\; 2\,\partial^\nu X_{\mu\nu}\,] \quad . \end{aligned} \tag{2.14}$$

which is *not* the same as the final line in (2.12). In any event we next use (2.12) to calculate the anti-commutator of the D-operator as realized on the fields of the double tensor multiplet and find

$$\begin{aligned} \{\,\mathrm{D}_a\,,\,\mathrm{D}_b\,\}\,X_{\mu\nu} &= i\,2\,(\gamma^\rho)_{a\,b}\,\partial_\rho\,X_{\mu\nu} \;+\; \partial_\mu\,s_{\nu\;a\,b} \;-\; \partial_\nu\,s_{\mu\;a\,b} \\ &\quad -\; i\,[\,\eta_{\alpha\,\mu}\,(\gamma_\nu)_{a\,b} \;-\; \eta_{\alpha\,\nu}\,(\gamma_\mu)_{a\,b}\,]\,\epsilon^{\alpha\,\rho\,\sigma\,\tau}\partial_\rho Y_{\sigma\,\tau} \;, \\ \{\,\mathrm{D}_a\,,\,\mathrm{D}_b\,\}\,Y_{\mu\nu} &= i\,2\,(\gamma^\rho)_{a\,b}\,\partial_\rho\,Y_{\mu\nu} \;+\; \partial_\mu\,t_{\nu\;a\,b} \;-\; \partial_\nu\,t_{\mu\;a\,b} \\ &\quad +\; i\,[\,\eta_{\alpha\,\mu}\,(\gamma_\nu)_{a\,b} \;-\; \eta_{\alpha\,\nu}\,(\gamma_\mu)_{a\,b}\,]\,\epsilon^{\alpha\,\rho\,\sigma\,\tau}\partial_\rho X_{\sigma\,\tau} \;, \\ s_{\mu\;a\,b} &\equiv i\,2\,(\gamma^\nu)_{a\,b}\,X_{\mu\nu} \;, \qquad t_{\mu\;a\,b} \;\equiv\; i\,2\,(\gamma^\nu)_{a\,b}\,Y_{\mu\nu} \;, \\ \{\,\mathrm{D}_a\,,\,\mathrm{D}_b\,\}\,\Lambda_c &= i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\,\Lambda_c \;+\; i\,(\gamma^\mu)_{a\,b}\,(\gamma_\mu\,\gamma^\nu)_c{}^d\,\partial_\nu\,\Lambda_d \quad . \end{aligned} \tag{2.15}$$

We can begin our analysis of (16) by concentrating on the anticommutator as realized on the fermion Λ_a. Similar to (6), we can write

$$
\begin{aligned}
\{\, \mathrm{D}_a \,, \mathrm{D}_b \,\} \Lambda_c \;&=\; i\, 2\, (\gamma^\mu)_{a\, b}\, \partial_\mu \Lambda_c \;+\; i\, 2\, (\gamma^\mu)_{a\, b}\, (\gamma_\mu)_c{}^d \widetilde{\mathcal{K}}_d(\Lambda) \;\;, \\
\widetilde{\mathcal{K}}_c(\Lambda) \;&=\; \frac{1}{2}\, (\gamma^\nu)_c{}^d \partial_\nu \Lambda_d \quad .
\end{aligned}
\tag{2.16}
$$

and we see the emergence of a non-closure function $\widetilde{\mathcal{K}}_c(\Lambda)$ as in the case of the on-shell chiral multiplet. This equation proves that the double tensor multiplet is an on-shell construction and no finite set of auxiliary fields is known to alleviate this.

The forms of the anticommutator as realized on $X_{\mu\nu}$ and $Y_{\mu\nu}$ reveal that any theory with the symmetries described by (2.12) must also possess translation symmetry and the gauge symmetries for both two-form fields. However, the last term of the first equation and the last term in the second equation in (16) also imply something new. These theories must also possess a symmetry under a 'Killing vector' of the form

$$
\delta_Z \;=\; -\, i2\, \xi_\mu\, \epsilon_\nu{}^{\rho\,\sigma\,\tau} \left[(\partial_\rho Y_{\sigma\,\tau}) \frac{\partial}{\partial X_{\mu\nu}} \;-\; (\partial_\rho X_{\sigma\,\tau}) \frac{\partial}{\partial Y_{\mu\nu}} \right] \quad . \tag{2.17}
$$

Since bosons typically satisfy second order differential equations of motion, this term (known as a 'central charge') *cannot* be interpreted as a non-closure term that vanishes upon use of the equations of motion. In fact, this is a 'on-shell central charge,' meaning that is has a non-trivial effect on the fields even when the theory obeys its equations of motion.

2.4 Review of 4D, $\mathcal{N} = 1$ Vector Multiplet

The 4D, $\mathcal{N} = 1$ vector multiplet off-shell is described by a vector A_μ, a Majorana fermion λ_a, and a pseudoscalar auxiliary field d. Their supersymmetry variations are described by

$$
\begin{aligned}
\mathrm{D}_a\, A_\mu \;&=\; (\gamma_\mu)_a{}^b\, \lambda_b \quad , \\
\mathrm{D}_a \lambda_b \;&=\; -\, i\, \frac{1}{4} ([\, \gamma^\mu \,, \gamma^\nu \,])_{ab}\, (\, \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \,) \;+\; (\gamma^5)_{a\, b}\, \mathrm{d} \;\;, \\
\mathrm{D}_a\, \mathrm{d} \;&=\; i\, (\gamma^5 \gamma^\mu)_a{}^b\, \partial_\mu \lambda_b \quad .
\end{aligned}
\tag{2.18}
$$

These lead in a straightforward manner to the following anticommutator algebra.

$$
\begin{aligned}
\{\, \mathrm{D}_a \,, \mathrm{D}_b \,\} A_\mu \;&=\; i\, 2\, (\gamma^\rho)_{a\, b}\, \partial_\rho A_\mu \;-\; \partial_\mu r_{a\, b} \;\;, \quad r_{a\, b} \;\equiv\; i\, 2\, (\gamma^\nu)_{a\, b}\, A_\nu \;\;, \\
\{\, \mathrm{D}_a \,, \mathrm{D}_b \,\} \lambda_c \;&=\; i\, 2\, (\gamma^\mu)_{a\, b}\, \partial_\mu \lambda_c \;\;, \\
\{\, \mathrm{D}_a \,, \mathrm{D}_b \,\} \mathrm{d} \;&=\; i\, 2\, (\gamma^\mu)_{a\, b}\, \partial_\mu \mathrm{d} \quad .
\end{aligned}
\tag{2.19}
$$

The term involving r_{ab} implies that any theory involving the vector gauge field above must also admit a symmetry of the form

$$
\delta_G^{(1)}(\alpha)\, A_\mu \;=\; \partial_\mu \alpha \quad , \tag{2.20}
$$

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which is easily identifiable as the usual form of a spin-1 gauge transformation.

In the on-shell theory, we set $d = 0$ but retain all other terms in (2.18)

$$\begin{aligned} \mathrm{D}_a\, A_\mu \;&=\; (\gamma^\mu)_a{}^b\, \lambda_b \quad , \\ \mathrm{D}_a \lambda_b \;&=\; -\, i\, \frac{1}{4} ([\,\gamma^\mu \,,\, \gamma^\nu\,])_{ab}\, (\,\partial_\mu\, A_\nu \;-\; \partial_\nu\, A_\mu\,) \quad , \end{aligned} \tag{2.21}$$

and once again we calculate the anticommutator as realized on the remaining fields to find

$$\begin{aligned} \{\, \mathrm{D}_a \,,\, \mathrm{D}_b \,\}\, A_\mu \;&=\; i\, 2\, (\gamma^\rho)_{a\,b}\, \partial_\rho\, A_\mu \;-\; \partial_\mu\, r_{a\,b} \;\;, \quad r_{a\,b} \;\equiv\; i\, 2\, (\gamma^\nu)_{a\,b}\, A_\nu \;\;, \\ \{\, \mathrm{D}_a \,,\, \mathrm{D}_b \,\}\, \lambda_c \;&=\; i\, 2\, (\gamma^\mu)_{a\,b}\, \partial_\mu\, \lambda_c \;-\; i\, \frac{1}{2}\, (\gamma^\mu)_{a\,b}\, (\gamma_\mu \gamma^\nu)_c{}^d\, \partial_\nu\, \lambda_d \\ &\quad +\; i\, \frac{1}{16}\, ([\,\gamma^\alpha \,,\, \gamma^\beta\,])_{a\,b}\, ([\,\gamma_\alpha \,,\, \gamma_\beta\,]\gamma^\nu)_c{}^d\, \partial_\nu\, \lambda_d \quad . \end{aligned} \tag{2.22}$$

The final equation of (2.22) shows the presence of *two* non-closure terms. We may rewrite the final line as

$$\begin{aligned} \{\, \mathrm{D}_a \,,\, \mathrm{D}_b \,\}\, \lambda_c \;&=\; i\, 2\, (\gamma^\mu)_{a\,b}\, \partial_\mu\, \lambda_c \;+\; i 2\, (\gamma^\mu)_{a\,b}\, (\gamma_\mu)_c{}^d\, \widehat{K}_d(\lambda) \\ &\quad -\; i\, \frac{1}{4}\, ([\,\gamma^\alpha \,,\, \gamma^\beta\,])_{a\,b}\, ([\,\gamma_\alpha \,,\, \gamma_\beta\,])_c{}^d\, \widehat{K}_d(\lambda) \;\;, \\ \widehat{K}_c(\lambda) \;&\equiv\; -\, \frac{1}{4} (\gamma^\nu)_c{}^d\, \partial_\nu\, \lambda_d \;=\; i\, \frac{1}{4}\, (\gamma^5)_c{}^d \mathrm{D}_d \mathrm{d} \;\;, \end{aligned} \tag{2.23}$$

where the non-closure term $\widehat{K}_c(\lambda)$ is introduced. Once more, it is seen to be consistent to set $\mathrm{d} = 0$ if the non-closure term vanishes, i.e. the fermion obeys an equation of motion.

3 Garden algebra matrices from 0-brane reduction

The four previous sections have presented a review of well known results. In this section, we will undertake to uncover the form of the Garden Algebra matrices associated with each supermultiplet previously discussed. According to the technique proposed in [10] this goal can be achieved by first performing a toroidal compactification of any higher D supersymmetircal multiplet on a 0-brane and thus retain only the temporal dependence of all fields in the supermultiplet.

3.1 4D, $\mathcal{N} = 1$ chiral multiplet on the 0-brane

The supersymmetry transformation laws in (2.1) are generally valid independent of the coordinate dependence of the various functions that appear in the equations. These equations remain valid if we restrict the functions so that they remain dependent only on the τ-coordinate. Under this restriction these equations can be recast in the form

$$\begin{array}{llll} \mathrm{D}_1 A = \;\; \psi_1 & \mathrm{D}_2 A = \;\psi_2 & \mathrm{D}_3 A = \;\; \psi_3 & \mathrm{D}_4 A = \;\psi_4 \\ \mathrm{D}_1 B = -\,\psi_4 & \mathrm{D}_2 B = \;\psi_3 & \mathrm{D}_3 B = -\psi_2 & \mathrm{D}_4 B = \;\psi_1 \\ \mathrm{D}_1 F = \;\; \partial_0 \psi_2 & \mathrm{D}_2 F = -\partial_0 \psi_1 & \mathrm{D}_3 F = -\partial_0 \psi_4 & \mathrm{D}_4 F = \;\partial_0 \psi_3 \\ \mathrm{D}_1 G = \;-\,\partial_0 \psi_3 & \mathrm{D}_2 G = -\partial_0 \psi_4 & \mathrm{D}_3 G = \;\; \partial_0 \psi_1 & \mathrm{D}_4 G = \;\partial_0 \psi_2 \quad . \end{array} \tag{3.1}$$

Next a set of re-definitions can be carried out on the fermions according to

$$\psi_1 \ \to \ i\,\Psi_1 \quad , \quad \psi_2 \ \to \ i\,\Psi_2 \quad , \quad \psi_3 \ \to \ i\,\Psi_3 \quad , \quad \psi_4 \ \to \ i\,\Psi_4 \quad , \tag{3.2}$$

so the previous equations take the forms of

$$\begin{array}{llll} \mathrm{D}_1 A = \ \ i\,\Psi_1 & \mathrm{D}_2 A = i\,\Psi_2 & \mathrm{D}_3 A = \ \ i\,\Psi_3 & \mathrm{D}_4 A = i\,\Psi_4 \\ \mathrm{D}_1 B = -i\,\Psi_4 & \mathrm{D}_2 B = \ i\,\Psi_3 & \mathrm{D}_3 B = -i\,\Psi_2 & \mathrm{D}_4 B = i\,\Psi_1 \\ \mathrm{D}_1 F = \ i\,\partial_0 \Psi_2 & \mathrm{D}_2 F = -i\partial_0\,\Psi_1 & \mathrm{D}_3 F = -i\partial_0\,\Psi_4 & \mathrm{D}_4 F = i\partial_0\,\Psi_3 \\ \mathrm{D}_1 G = \ -i\partial_0 \Psi_3 & \mathrm{D}_2 G = -i\partial_0\,\Psi_4 & \mathrm{D}_3 G = \ \ i\partial_0\,\Psi_1 & \mathrm{D}_4 G = i\partial_0\,\Psi_2 \ . \end{array} \tag{3.3}$$

Now we define

$$\Phi_1 \ = \ A \quad , \quad \Phi_2 \ = \ B \quad , \quad \partial_0 \Phi_3 \ = \ F \quad , \quad \partial_0 \Phi_4 \ = \ G \quad , \tag{3.4}$$

and note the above system of equations can be written in the form

$$\mathrm{D}_{\mathrm{I}} \Phi_i \ = \ i\,(\mathrm{L}_{\mathrm{I}})_{i\,\hat{k}}\,\Psi_{\hat{k}} \quad . \tag{3.5}$$

The explicit form of the L-matrices that appear here are given by

$$(\mathrm{L}_1)_{i\,\hat{k}} \ = \ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} , \qquad (\mathrm{L}_2)_{i\,\hat{k}} \ = \ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} ,$$

$$(\mathrm{L}_3)_{i\,\hat{k}} \ = \ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} , \qquad (\mathrm{L}_4)_{i\,\hat{k}} \ = \ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} . \tag{3.6}$$

After writing the results for the fermions we find

$$\begin{array}{llll} \mathrm{D}_1 \Psi_1 = \partial_0 A & \mathrm{D}_2 \Psi_1 = -F & \mathrm{D}_3 \Psi_1 = \ \ G & \mathrm{D}_4 \Psi_1 = \partial_0 B \\ \mathrm{D}_1 \Psi_2 = \ \ F & \mathrm{D}_2 \Psi_2 = \partial_0 A & \mathrm{D}_3 \Psi_2 = -\partial_0 B & \mathrm{D}_4 \Psi_2 = G \\ \mathrm{D}_1 \Psi_3 = -G & \mathrm{D}_2 \Psi_3 = \ \partial_0 B & \mathrm{D}_3 \Psi_3 = \ \ \partial_0 A & \mathrm{D}_4 \Psi_3 = F \\ \mathrm{D}_1 \Psi_4 = -\partial_0 B & \mathrm{D}_2 \Psi_4 = \ -G & \mathrm{D}_3 \Psi_4 = \ \ -F & \mathrm{D}_4 \Psi_4 = \partial_0 A \ . \end{array} \tag{3.7}$$

Once more we use the definitions in (3.4) and note that the above system of equations can be written in the form

$$\mathrm{D}_{\mathrm{I}} \Psi_{\hat{k}} \ = \ (\mathrm{R}_{\mathrm{I}})_{\hat{k}\,i}\,\frac{d}{dt}\,\Phi_i \quad . \tag{3.8}$$

The explicit form of the matrices that appear here are given by

$$(\mathrm{R}_1)_{i\,\hat{k}} \ = \ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix} , \quad (\mathrm{R}_2)_{i\,\hat{k}} \ = \ \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} ,$$

$$(\mathrm{R}_3)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} , \quad (\mathrm{R}_4)_{i\hat{k}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} . \tag{3.9}$$

It is now seen that the set of L-matrices (3.6) and R-matrices (3.9) satisfy the equation

$$(\mathrm{R}_{\mathrm{I}}) \equiv [(\mathrm{L}_{\mathrm{I}})]^t \tag{3.10}$$

where the t superscript stands for transposition.

We now turn our attention to the on-shell case. This begins by setting $F = G = 0$. The consistency of these conditions implies $\partial_0 \psi_{\hat{k}} = \partial_0^2 A = \partial_0^2 B = 0$. Further consistency conditions imply that Φ_i be defined by

$$\Phi_1 = A \ , \quad \Phi_2 = B \ , \tag{3.11}$$

while $\Psi_{\hat{k}}$ is still defined by (3.2). In these considerations of the on-shell theory, (3.5) and (3.7) are still valid. However, the definition of the L-matrices and R-matrices must now be changed to

$$\begin{aligned}
(\mathrm{L}_1)_{i\hat{k}} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad & (\mathrm{L}_2)_{i\hat{k}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \\
(\mathrm{L}_3)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad & (\mathrm{L}_4)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} , \\
(\mathrm{R}_1)_{i\hat{k}} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad & (\mathrm{R}_2)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} ,
\end{aligned} \tag{3.12}$$

$$(\mathrm{R}_3)_{i\hat{k}} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (\mathrm{R}_4)_{i\hat{k}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} . \tag{3.13}$$

3.2 4D, $\mathcal{N} = 1$ tensor multiplet on the 0-brane

We now repeat the process of the last subsection. However, we now take as our starting point the results in (2.7). Carrying out the reduction yields the following for the bosons

$$\begin{array}{llll}
\mathrm{D}_1\phi = \chi_1 & \mathrm{D}_2\phi = \chi_2 & \mathrm{D}_3\phi = \chi_3 & \mathrm{D}_4\phi = \chi_4 \\
2\,\mathrm{D}_1 B_{12} = -\chi_3 & 2\,\mathrm{D}_2 B_{12} = \chi_4 & 2\,\mathrm{D}_3 B_{12} = \chi_1 & 2\,\mathrm{D}_4 B_{12} = -\chi_2 \\
2\,\mathrm{D}_1 B_{23} = -\chi_4 & 2\,\mathrm{D}_2 B_{23} = -\chi_3 & 2\,\mathrm{D}_3 B_{23} = \chi_2 & 2\,\mathrm{D}_4 B_{23} = \chi_1 \\
2\,\mathrm{D}_1 B_{31} = -\chi_2 & 2\,\mathrm{D}_2 B_{31} = \chi_1 & 2\,\mathrm{D}_3 B_{31} = -\chi_4 & 2\,\mathrm{D}_4 B_{31} = \chi_3 \ ,
\end{array} \tag{3.14}$$

and for the fermions the analogous results,

$$\begin{array}{llll}
\mathrm{D}_1\chi_1 = i\partial_0\phi & \mathrm{D}_2\chi_1 = i2\partial_0 B_{31} & \mathrm{D}_3\chi_1 = i2\partial_0 B_{12} & \mathrm{D}_4\chi_1 = i2\partial_0 B_{23} \\
\mathrm{D}_1\chi_2 = -i2\partial_0 B_{31} & \mathrm{D}_2\chi_2 = i\partial_0\phi & \mathrm{D}_3\chi_2 = i2\partial_0 B_{23} & \mathrm{D}_4\chi_2 = -i2\partial_0 B_{12} \\
\mathrm{D}_1\chi_3 = -2\partial_0 B_{12} & \mathrm{D}_2\chi_3 = -2\partial_0 B_{23} & \mathrm{D}_3\chi_3 = i\partial_0\phi & \mathrm{D}_4\chi_3 = i2\partial_0 B_{31} \\
\mathrm{D}_1\chi_4 = -i2\partial_0 B_{23} & \mathrm{D}_2\chi_4 = i2\partial_0 B_{12} & \mathrm{D}_3\chi_4 = -i2\partial_0 B_{31} & \mathrm{D}_4\chi_4 = i\partial_0\phi
\end{array} . \tag{3.15}$$

Next the fermions are re-defined according to

$$\chi_1 \;\to\; i\,\Psi_1 \quad , \quad \chi_2 \;\to\; i\,\Psi_2 \quad , \quad \chi_3 \;\to\; i\,\Psi_3 \quad , \quad \chi_4 \;\to\; i\,\Psi_4 \tag{3.16}$$

and the bosons are re-defined according to

$$\Phi_1 \;=\; \phi \quad , \quad \Phi_2 \;=\; 2\,B_{12} \quad , \quad \Phi_3 \;=\; 2\,B_{23} \quad , \quad \Phi_4 \;=\; 2\,B_{31} \;, \tag{3.17}$$

so the above system of equations ((3.14) and (3.15)) respectively can be written in the forms

$$\mathrm{D}_{\mathrm{I}}\Phi_i \;=\; i\;(\mathrm{L}_{\mathrm{I}})_{\,i\,\hat{k}}\,\Psi_{\hat{k}} \;\; , \;\; \mathrm{D}_{\mathrm{I}}\Psi_{\hat{k}} \;=\; (\mathrm{R}_{\mathrm{I}})_{\,\hat{k}\,i}\,\frac{d}{dt}\,\Phi_i \tag{3.18}$$

where the explicit form of the L-matrices that appear here are given by

$$(\mathrm{L}_1)_{\,i\,\hat{k}} \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix} , \qquad (\mathrm{L}_2)_{\,i\,\hat{k}} \;=\; \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} ,$$
$$(\mathrm{L}_3)_{\,i\,\hat{k}} \;=\; \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} , \qquad (\mathrm{L}_4)_{\,i\,\hat{k}} \;=\; \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \tag{3.19}$$

and

$$(\mathrm{R}_1)_{\,i\,\hat{k}} \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} , \qquad (\mathrm{R}_2)_{\,i\,\hat{k}} \;=\; \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} ,$$
$$(\mathrm{R}_3)_{\,i\,\hat{k}} \;=\; \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} , \qquad (\mathrm{R}_4)_{\,i\,\hat{k}} \;=\; \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} . \tag{3.20}$$

There is also another feature that is noticable from (2.7). It is clear that we also obtain

$$\begin{aligned}
\mathrm{D}_1 B_{0\,1} \;&=\; \frac{1}{2}\chi_1 \quad , \;\; \mathrm{D}_1 B_{0\,2} \;=\; \frac{1}{2}\chi_3 \;, \;\; \mathrm{D}_1 B_{0\,3} \;=\; \frac{1}{2}\chi_2 \\
\mathrm{D}_2 B_{0\,1} \;&=\; \frac{1}{2}\chi_2 \quad , \;\; \mathrm{D}_2 B_{0\,2} \;=\; \frac{1}{2}\chi_4 \;, \;\; \mathrm{D}_2 B_{0\,3} \;=\; \frac{1}{2}\chi_2 \\
\mathrm{D}_2 B_{0\,1} \;&=\; -\,\frac{1}{2}\chi_3 \;, \;\; \mathrm{D}_3 B_{0\,2} \;=\; \frac{1}{2}\chi_1 \;, \;\; \mathrm{D}_3 B_{0\,3} \;=\; -\,\frac{1}{2}\chi_4 \\
\mathrm{D}_4 B_{0\,1} \;&=\; -\,\frac{1}{2}\chi_4 \;, \;\; \mathrm{D}_4 B_{0\,2} \;=\; \frac{1}{2}\chi_2 \;, \;\; \mathrm{D}_4 B_{0\,3} \;=\; -\,\frac{1}{2}\chi_3 \;,
\end{aligned} \tag{3.21}$$

in addition to the results in (3.14). However, on the right hand side of the equations in (3.15) there are *no* appearances of terms that depend on $B_{0\,1}$, $B_{0\,2}$, or $B_{0\,3}$. From (2.10) it can be seen that

$$\delta_G^{(2)} B_{0\,1} \;=\; \partial_0\,\ell_1 \;\; , \;\; \delta_G^{(2)} B_{0\,2} \;=\; \partial_0\,\ell_2 \;\; , \;\; \delta_G^{(2)} B_{0\,3} \;=\; \partial_0\,\ell_3 \tag{3.22}$$

expresses the form of the gauge transformation on the 0-brane. The two equations (3.21) and (3.22) together imply that:

(a.) the gauge transformations make it possible to choose a gauge where $B_{0\,1} = B_{0\,2} = B_{0\,3} = 0$ (ignoring issues related to zero-modes),

(b.) the supersymmetry variations described by (3.21) take one out of this gauge, and

(c.) there exist further gauge transformations that can be used to restore the $B_{0\,1} = B_{0\,2} = B_{0\,3} = 0$ gauge condition.

So on the 0-brane it is consistent to simply ignore the field components ($B_{0\,1}$, $B_{0\,2}$, $B_{0\,3}$) and work in a 'Coulomb gauge'.

3.3 4D, $\mathcal{N} = 1$ Double Tensor Multiplet On The 0-Brane

Starting from (2.12) we find carrying out the reduction for the bosons leads to

$$\begin{array}{ll}
\mathrm{D}_1 X_{12} = -(\tfrac{1}{2}\Lambda_2) & \mathrm{D}_2 X_{12} = -(\tfrac{1}{2}\Lambda_1) \\
\mathrm{D}_1 X_{23} = +(\tfrac{1}{2}\Lambda_1) & \mathrm{D}_2 X_{23} = -(\tfrac{1}{2}\Lambda_2) \\
\mathrm{D}_1 X_{31} = +(\tfrac{1}{2}\Lambda_3) & \mathrm{D}_2 X_{31} = +(\tfrac{1}{2}\Lambda_4) \\
\mathrm{D}_1 Y_{12} = -(\tfrac{1}{2}\Lambda_3) & \mathrm{D}_2 Y_{12} = +(\tfrac{1}{2}\Lambda_4) \\
\mathrm{D}_1 Y_{23} = -(\tfrac{1}{2}\Lambda_4) & \mathrm{D}_2 Y_{23} = -(\tfrac{1}{2}\Lambda_3) \\
\mathrm{D}_1 Y_{31} = -(\tfrac{1}{2}\Lambda_2) & \mathrm{D}_2 Y_{31} = +(\tfrac{1}{2}\Lambda_1)
\end{array} \tag{3.23}$$

$$\begin{array}{ll}
\mathrm{D}_3 X_{12} = +(\tfrac{1}{2}\Lambda_4) & \mathrm{D}_4 X_{12} = +(\tfrac{1}{2}\Lambda_3) \\
\mathrm{D}_3 X_{23} = -(\tfrac{1}{2}\Lambda_3) & \mathrm{D}_4 X_{23} = +(\tfrac{1}{2}\Lambda_2) \\
\mathrm{D}_3 X_{31} = +(\tfrac{1}{2}\Lambda_1) & \mathrm{D}_4 X_{31} = +(\tfrac{1}{2}\Lambda_2) \\
\mathrm{D}_3 Y_{12} = +(\tfrac{1}{2}\Lambda_1) & \mathrm{D}_4 Y_{12} = -(\tfrac{1}{2}\Lambda_2) \\
\mathrm{D}_3 Y_{23} = +(\tfrac{1}{2}\Lambda_2) & \mathrm{D}_4 Y_{23} = +(\tfrac{1}{2}\Lambda_1) \\
\mathrm{D}_3 Y_{31} = -(\tfrac{1}{2}\Lambda_4) & \mathrm{D}_4 Y_{31} = +(\tfrac{1}{2}\Lambda_3)
\end{array} \tag{3.24}$$

and for the fermions

$$\begin{array}{ll}
\mathrm{D}_1\,(\tfrac{1}{2}\,\Lambda_1) = +i\,\partial_0 X_{23} & \mathrm{D}_2\,(\tfrac{1}{2}\,\Lambda_1) = -i\,\partial_0 X_{12} + i\,\partial_0 Y_{31} \\
\mathrm{D}_1\,(\tfrac{1}{2}\,\Lambda_2) = -i\,\partial_0 X_{12} - i\,\partial_0 Y_{31} & \mathrm{D}_2\,(\tfrac{1}{2}\,\Lambda_2) = -i\,\partial_0 X_{23} \\
\mathrm{D}_1\,(\tfrac{1}{2}\,\Lambda_3) = +i\,\partial_0 X_{31} - i\,\partial_0 Y_{12} & \mathrm{D}_2\,(\tfrac{1}{2}\,\Lambda_3) = -i\,\partial_0 Y_{23} \\
\mathrm{D}_1\,(\tfrac{1}{2}\,\Lambda_4) = -i\,\partial_0 Y_{23} & \mathrm{D}_2\,(\tfrac{1}{2}\,\Lambda_4) = +i\,\partial_0 X_{31} + i\,\partial_0 Y_{12}
\end{array} \tag{3.25}$$

$$\begin{array}{ll}
\mathrm{D}_3\,(\tfrac{1}{2}\,\Lambda_1) = +i\,\partial_0 X_{31} + i\,\partial_0 Y_{12} & \mathrm{D}_4\,(\tfrac{1}{2}\,\Lambda_1) = +i\,\partial_0 Y_{23} \\
\mathrm{D}_3\,(\tfrac{1}{2}\,\Lambda_2) = +i\,\partial_0 Y_{23} & \mathrm{D}_4\,(\tfrac{1}{2}\,\Lambda_2) = +i\,\partial_0 X_{31} - i\,\partial_0 Y_{12} \\
\mathrm{D}_3\,(\tfrac{1}{2}\,\Lambda_3) = -i\,\partial_0 X_{23} & \mathrm{D}_4\,(\tfrac{1}{2}\,\Lambda_3) = +i\,\partial_0 X_{12} + i\,\partial_0 Y_{31} \\
\mathrm{D}_3\,(\tfrac{1}{2}\,\Lambda_4) = +i\,\partial_0 X_{12} - i\,\partial_0 Y_{31} & \mathrm{D}_4\,(\tfrac{1}{2}\,\Lambda_4) = +i\,\partial_0 X_{23}
\end{array} \quad . \tag{3.26}$$

Using the notation:

$$\Phi_i \;=\; (X_{12}, \quad X_{23}, \quad X_{31}, \quad Y_{12}, \quad Y_{23}, \quad Y_{31}) \;\; , \tag{3.27}$$

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and for the fermions

$$\frac{1}{2}\Lambda_1 \to i\,\Psi_1 \quad , \quad \frac{1}{2}\Lambda_2 \to i\,\Psi_2 \quad , \quad \frac{1}{2}\Lambda_3 \to i\,\Psi_3 \quad , \quad \frac{1}{2}\Lambda_4 \to i\,\Psi_4 \ , \tag{3.28}$$

the above systems of equations can be written in the form of (3.5) and (3.8). The explicit form of the matrices that appear here are given by

$$(\mathrm{L}_1)_{i\hat{k}} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad (\mathrm{L}_2)_{i\hat{k}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$(\mathrm{L}_3)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad (\mathrm{L}_4)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.29}$$

$$(\mathrm{R}_1)_{i\hat{k}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$(\mathrm{R}_{12})_{i\hat{k}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix},$$

$$(\mathrm{R}_3)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$(\mathrm{R}_4)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.30}$$

3.4 4D, $\mathcal{N} = 1$ vector multiplet on the 0-brane

Starting from (2.18) we find that carrying out the reduction for the bosons leads to

$$\begin{array}{llll} \mathrm{D}_1 A_1 = \lambda_2 & \mathrm{D}_2 A_1 = \lambda_1 & \mathrm{D}_3 A_1 = \lambda_4 & \mathrm{D}_4 A_1 = \lambda_3 \\ \mathrm{D}_1 A_2 = -\lambda_4 & \mathrm{D}_2 A_2 = \lambda_3 & \mathrm{D}_3 A_2 = \lambda_2 & \mathrm{D}_4 A_2 = -\lambda_1 \\ \mathrm{D}_1 A_3 = \lambda_1 & \mathrm{D}_2 A_3 = -\lambda_2 & \mathrm{D}_3 A_3 = \lambda_3 & \mathrm{D}_4 A_3 = -\lambda_4 \\ \mathrm{D}_1 \mathrm{d} = -\partial_0 \lambda_3 & \mathrm{D}_2 \mathrm{d} = -\partial_0 \lambda_4 & \mathrm{D}_3 \mathrm{d} = \partial_0 \lambda_1 & \mathrm{D}_4 \mathrm{d} = \partial_0 \lambda_2 \end{array} \tag{3.31}$$

and for the fermions

$$
\begin{array}{llll}
\mathrm{D}_1\lambda_1 = \ \ i\partial_0 A_3 & \mathrm{D}_2\lambda_1 = \ \ i\partial_0 A_1 & \mathrm{D}_3\lambda_1 = \ \ i\,\mathrm{d} & \mathrm{D}_4\lambda_1 = -i\partial_0 A_2 \\
\mathrm{D}_1\lambda_2 = \ \ i\partial_0 A_1 & \mathrm{D}_2\lambda_2 = -i\partial_0 A_3 & \mathrm{D}_3\lambda_2 = i\partial_0 A_2 & \mathrm{D}_4\lambda_2 = \ \ i\,\mathrm{d} \\
\mathrm{D}_1\lambda_3 = \ \ -i\mathrm{d} & \mathrm{D}_2\lambda_3 = \ \ i\partial_0 A_2 & \mathrm{D}_3\lambda_3 = i\partial_0 A_3 & \mathrm{D}_4\lambda_3 = \ \ i\partial_0 A_1 \\
\mathrm{D}_1\lambda_4 = -i\partial_0 A_2 & \mathrm{D}_2\lambda_4 = \ \ -i\mathrm{d} & \mathrm{D}_3\lambda_4 = i\partial_0 A_1 & \mathrm{D}_4\lambda_4 = -i\partial_0 A_3
\end{array}
\tag{3.32}
$$

so these suggest the following identifications for the Φ's and Ψ's

$$
\lambda_1 \rightarrow i\,\Psi_1 \quad , \quad \lambda_2 \rightarrow i\,\Psi_2 \quad , \quad \lambda_3 \rightarrow i\,\Psi_3 \quad , \quad \lambda_4 \rightarrow i\,\Psi_4 \quad , \tag{3.33}
$$

$$
\Phi_1 \ = \ A_1 \quad , \quad \Phi_2 \ = \ A_2 \quad , \quad \Phi_3 \ = \ A_3 \quad , \quad \partial_0 \Phi_4 \ = \ \mathrm{d} \quad . \tag{3.34}
$$

We continue as in the previous discussion to define the L-matrices and R-matrices. Given the equations (3.31)–(3.34) we find the results below for the L-matrices

$$
(\mathrm{L}_1)_{i\,\hat{k}} \ = \ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} , \qquad (\mathrm{L}_2)_{i\,\hat{k}} \ = \ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} ,
$$

$$
(\mathrm{L}_3)_{i\,\hat{k}} \ = \ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} , \qquad (\mathrm{L}_4)_{i\,\hat{k}} \ = \ \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} , \tag{3.35}
$$

and the associated R-matrices are found by the relation in (3.7).

Similarity to the case of the tensor multiplet can also be seen. In addition to the results in (3.31) we also have

$$
\mathrm{D}_1 A_0 \ = \ - \ \lambda_2 \ , \quad \mathrm{D}_2 A_0 \ = \ \lambda_1 \ , \quad \mathrm{D}_3 A_0 \ = \ \lambda_4 \ , \quad \mathrm{D}_4 A_0 \ = \ - \ \lambda_3 \ . \ , \tag{3.36}
$$

Furthermore, there is no appearance of A_0 in the equations of (3.32) and there is the gauge transformation as stated in (2.20). Thus it is consistent to work in the Coulomb gauge where we set $A_0 = 0$ throughout our considerations of the vector multiplet.

As with the chiral multiplet, it is possible in the case of the vector multiplet to consider the on-shell theory. This begins by setting $d = 0$. The consistency of these conditions imply $\partial_0 \lambda_{\hat{k}} = \partial_0^2 A_1 = \partial_0^2 A_2 = \partial_0^2 A_3 = 0$. Further consistency conditions implies that Φ_i be defined by

$$
\Phi_1 \ = \ A_1 \quad , \quad \Phi_2 \ = \ A_2 \quad , \quad \Phi_3 \ = \ A_3 \quad , \tag{3.37}
$$

while the Ψ-fermions are still defined by (3.33). Using these definitions, the on-shell vector multiplet satisfies equations as in (3.18), but with the L-matrices define by

$$
(\mathrm{L}_1)_{i\,\hat{k}} \ = \ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad , \quad (\mathrm{L}_2)_{i\,\hat{k}} \ = \ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad ,
$$

$$(\mathrm{L}_3)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad , \quad (\mathrm{L}_4)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad , \tag{3.38}$$

and the R-matrices are found to satisfy (3.10).

3.5 Summary of multiplet reduction on the 0-brane

Earlier in this section, the Garden Algebra matrices associated with four 4D, $\mathcal{N} = 1$ supermultiplets were derived for:

(a.) the off-shell chiral multiplet where the associated L-matrices and R-matrices appear in (3.6) and (3.9) (case I),

(b.) the on-shell chiral multiplet where the associated L-matrices and R-matrices appear in (3.12) and (3.13) (case II),

(c.) the off-shell tensor multiplet where the associated L-matrices and R-matrices appear in (3.19) and (3.20) (case III),

(d.) the double tensor multiplet where the associated L-matrices and R-matrices appear in (3.29) and (3.30) (case IV),

(e.) the off-shell vector multiplet where the associated L-matrices and R-matrices appear in (3.35) and (3.8) (case V),

(f.) and the on-shell vector multiplet where the associated L-matrices and R-matrices appear in (3.38) and (3.8) (case VI).

For later convenience we will refer to these as case I through case VI.

Before the reduction procedure that reveals the matrices, the multiplets describe four 1D, $\mathcal{N} = 4$ theories. The matrices associated with each multiplet in the cases of I, III and V (the off-shell representations) share some common features. They all satisfy the equations

$$\begin{aligned} (\mathrm{L}_{\mathrm{I}})_i{}^{\hat{\jmath}}\,(\mathrm{R}_{\mathrm{J}})_{\hat{\jmath}}{}^k + (\mathrm{L}_{\mathrm{J}})_i{}^{\hat{\jmath}}\,(\mathrm{R}_{\mathrm{I}})_{\hat{\jmath}}{}^k &= 2\,\delta_{\mathrm{IJ}}\,\delta_i{}^k \ , \\ (\mathrm{R}_{\mathrm{J}})_{\hat{\imath}}{}^j\,(\mathrm{L}_{\mathrm{I}})_j{}^{\hat{k}} + (\mathrm{R}_{\mathrm{I}})_{\hat{\imath}}{}^j\,(\mathrm{L}_{\mathrm{J}})_j{}^{\hat{k}} &= 2\,\delta_{\mathrm{IJ}}\,\delta_{\hat{\imath}}{}^{\hat{k}} \ . \end{aligned} \tag{3.39}$$

$$(\mathrm{R}_{\mathrm{I}})_{\hat{\jmath}}{}^k\,\delta_{ik} = (\mathrm{L}_{\mathrm{I}})_i{}^{\hat{k}}\,\delta_{\hat{\jmath}\hat{k}} \ , \tag{3.40}$$

which we have named as the "$\mathcal{GR}$(d, $\mathcal{N}$) Algebras" or "Garden Algebras." Here the indices have ranges that correspond to I, J, etc. = 1, ..., $\mathcal{N}$, i, j, etc. = 1, ..., d_L, and $\hat{\imath}$, $\hat{\jmath}$, etc. = 1, ..., d_R for some integers $\mathcal{N}$, d_L, and d_R.

Throughout most previous discussions, there has only been consideration of the case where $\mathrm{d}_L = \mathrm{d}_R = \mathrm{d}$. In this case, the L-matrices and R-matrices may be assembled according to

$$\gamma_{\mathrm{I}} = \begin{bmatrix} 0 & \mathrm{L}_{\mathrm{I}} \\ \mathrm{R}_{\mathrm{I}} & 0 \end{bmatrix} \tag{3.41}$$

and we may introduce one additional 2d $\times$ 2d matrix $(-1)^{\mathcal{F}}$ where

$$(-1)^{\mathcal{F}} \; = \; \begin{bmatrix} \mathrm{I} & 0 \\ 0 & -\,\mathrm{I} \end{bmatrix} \quad . \tag{3.42}$$

Thus, due to (3.39), the $\gamma_{\rm I}$'s together with $(-1)^{\mathcal{F}}$ satisfy the Clifford Algebra $Cl(\mathcal{N}+1)$ over the reals.

However, the case where $\mathrm{d}_L \neq \mathrm{d}_R$ (i.e. cases II, IV and VI), can also be considered. In this more general case, the matrices may be described as belonging to a mathematical structure denoted by the symbol $\mathcal{GR}(\mathrm{d}_L,\, \mathrm{d}_R,\, \mathcal{N})$. In the case of on-shell theories, it is the case that $\mathrm{d}_L \neq \mathrm{d}_R$ so in order to extend the discussion of the previous works to the on-shell cases, we will have to consider $\mathcal{GR}(\mathrm{d}_L,\, \mathrm{d}_R,\, \mathcal{N})$ matrices for the on-shell theories as well as the Double Tensor Multiplet. We should add that since we have not studied $\mathcal{GR}(\mathrm{d}_L,\, \mathrm{d}_R,\, \mathcal{N})$, its precise nature is not understood. However, calculations involving this structure will be presented in an appendix.

4 Considering some traces

As we have seen from the discussions of the previous sections, each supersymmetrical multiplet has an associated set of L-matrices and R-matrices that are revealed upon reduction on a 0-brane. In general, however, these matrices do not have to be square. What we have shown is that that when the supermultiplet is off-shell, the matrices will be square. A question that might be interesting to consider is, "For a given multiplet, how unique are such matrices?"

Clearly, to obtain the matrices, we have made many arbitrary choices along the way. So the uniqueness question can also be cast in as the following form. Let us begin with the assumption that there exists two sets (linearly independent of one another) of real matrices such that $\mathrm{L}_{\rm I}$ and $\widehat{\mathrm{L}}_{\rm I}$ that satisfy[1]

$$\mathrm{L}_{\rm I}(\mathrm{L}_{\rm I})^t \;=\; (\mathrm{L}_{\rm I})^t\mathrm{L}_{\rm I} \;=\; \mathbf{I} \quad , \quad \widehat{\mathrm{L}}_{\rm I}(\widehat{\mathrm{L}}_{\rm I})^t \;=\; (\widehat{\mathrm{L}}_{\rm I})^t\widehat{\mathrm{L}}_{\rm I} \;=\; \mathbf{I} \quad . \tag{4.1}$$

$$\mathrm{L}_{\rm I}(\mathrm{L}_{\rm J})^t \;+\; \mathrm{L}_{\rm J}(\mathrm{L}_{\rm I})^t \;=\; 0 \quad , \quad (\widehat{\mathrm{L}}_{\rm I})^t\widehat{\mathrm{L}}_{\rm J} \;+\; (\widehat{\mathrm{L}}_{\rm J})^t\widehat{\mathrm{L}}_{\rm I} \;=\; 0 \quad . \tag{4.2}$$

We say that $\mathrm{L}_{\rm I}$ and $\widehat{\mathrm{L}}_{\rm I}$ are members of the same equivalence class if there exists real square matrices $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\widehat{\mathrm{L}}_{\rm I} \;=\; \mathcal{X}\,\mathrm{L}_{\rm I}\mathcal{Y} \tag{4.3}$$

and where

$$\mathcal{X}\,(\mathcal{X})^t \;=\; (\mathcal{X})^t\,\mathcal{X} \;=\; \mathcal{Y}\,(\mathcal{Y})^t \;=\; (\mathcal{Y})^t\,\mathcal{Y} \;=\; \mathbf{I} \quad . \tag{4.4}$$

[1]No summations over indices are implied for the equations in (4.1).

These last equations imply that $\mathcal{X}$ is an element of the $\mathrm{O}(\mathrm{d}_L)$ group while $\mathcal{Y}$ is an element of the $\mathrm{O}(\mathrm{d}_R)$ group.[2] Using (4.3), we next observe that

$$
\begin{aligned}
\widehat{\mathrm{L}}_{\mathrm{I}}(\widehat{\mathrm{L}}_{\mathrm{J}})^t &= \mathcal{X}\,[\,\mathrm{L}_{\mathrm{I}}(\mathrm{L}_{\mathrm{J}})^t\,]\,(\mathcal{X})^t \quad , \\
(\widehat{\mathrm{L}}_{\mathrm{I}})^t\widehat{\mathrm{L}}_{\mathrm{J}} &= (\mathcal{Y})^t\,[\,(\mathrm{L}_{\mathrm{I}})^t\mathrm{L}_{\mathrm{J}}\,]\,\mathcal{Y} \quad ,
\end{aligned}
\tag{4.5}
$$

or on taking traces we see

$$
\begin{aligned}
\mathrm{Tr}\left[\,\widehat{\mathrm{L}}_{\mathrm{I}_1}(\widehat{\mathrm{L}}_{\mathrm{J}_1})^t\,\right] &= \mathrm{Tr}\left[\,\mathrm{L}_{\mathrm{I}_1}(\mathrm{L}_{\mathrm{J}_1})^t\,\right] \quad , \\
\mathrm{Tr}\left[\,(\widehat{\mathrm{L}}_{\mathrm{I}_1})^t\widehat{\mathrm{L}}_{\mathrm{J}_1}\,\right] &= \mathrm{Tr}\left[\,(\mathrm{L}_{\mathrm{I}_1})^t\mathrm{L}_{\mathrm{J}_1}\,\right] \quad .
\end{aligned}
\tag{4.6}
$$

This property is shared by more general expressions of the form

$$
\begin{aligned}
\varphi^{(p)}{}_{\mathrm{I}_1\mathrm{J}_1\cdots\mathrm{I}_p\mathrm{J}_p} &= \mathrm{Tr}\left[\,\mathrm{L}_{\mathrm{I}_1}(\mathrm{L}_{\mathrm{J}_1})^t\cdots\,\mathrm{L}_{\mathrm{I}_p}(\mathrm{L}_{\mathrm{J}_p})^t\,\right] \quad , \\
\widetilde{\varphi}^{(p)}{}_{\mathrm{I}_1\mathrm{J}_1\cdots\mathrm{I}_p\mathrm{J}_p} &= \mathrm{Tr}\left[\,(\mathrm{L}_{\mathrm{I}_1})^t\mathrm{L}_{\mathrm{J}_1}\cdots\,(\mathrm{L}_{\mathrm{I}_p})^t\mathrm{L}_{\mathrm{J}_p}\,\right] \quad .
\end{aligned}
\tag{4.7}
$$

We note that for the present case under consideration, we will not consider $p > 2$. Furthermore, using the cyclicity of the trace operation we have

$$
\widetilde{\varphi}^{(p)}{}_{\mathrm{I}_1\mathrm{J}_1\cdots\mathrm{I}_p\mathrm{J}_p} = \varphi^{(p)}{}_{\mathrm{J}_1\mathrm{I}_2\cdots\mathrm{J}_p\mathrm{I}_1} \quad .
\tag{4.8}
$$

The collection of all such objects shares some of the properties of characters as for groups. Due to the identities in (4.5) the value of these objects is independent of the linear field redefinitions that leave a quadratic super-invariant (see [17]) unchanged. We will call these "chromocharacters" because their values still depend on the choices made to describe the supersymmetry generators. So these objects still depend on how the colors in an Adinkra are picked.

Since we have derived the L-matrices and R-matrices for six distinct cases, I, II, III, IV, V, and VI (as delineated in above equation (3.39)), we will denote the distinct cases by including a roman numeral after the symbol for the chromocharacter. Our calculations reveal

$$
\begin{aligned}
\varphi^{(1)}{}_{\mathrm{IJ}}(I) &= 4\,\delta_{\mathrm{IJ}} \quad , \\
\varphi^{(1)}{}_{\mathrm{IJ}}(II) &= 2\,\delta_{\mathrm{IJ}} \quad , \\
\varphi^{(1)}{}_{\mathrm{IJ}}(III) &= 4\,\delta_{\mathrm{IJ}} \quad , \\
\varphi^{(1)}{}_{\mathrm{IJ}}(IV) &= 6\,\delta_{\mathrm{IJ}} \quad , \\
\varphi^{(1)}{}_{\mathrm{IJ}}(V) &= 4\,\delta_{\mathrm{IJ}} \quad , \\
\varphi^{(1)}{}_{\mathrm{IJ}}(VI) &= 3\,\delta_{\mathrm{IJ}} \quad .
\end{aligned}
\tag{4.9}
$$

[2] The curious reader may well ask, "Why are these groups relevant?" Some insight into this comes from the work of [17]. There it was shown that for valise Adinkras, it is always possible to construct a supersymmetrical invariant that is quadratic in the fields of the Adinkra. There are a large set of linear field redefinitions that do *not* mix bosons and fermions, under which this supersymmetrical invariant remains unchanged. The $\mathrm{O}(\mathrm{d}_L)$ and $\mathrm{O}(\mathrm{d}_R)$ groups are related to these symmetries.

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The behavior of the $p = 2$ chromocharacters is very different for the off-shell cases (I, II, V) versus on-shell cases (II, IV, VI). We present the off-shell cases first:

$$
\begin{aligned}
\varphi^{(2)}{}_{\rm IJKL}(I) &= 4 \left[\delta_{\rm IJ}\delta_{\rm KL} - \delta_{\rm IK}\delta_{\rm JL} + \delta_{\rm IL}\delta_{\rm JK} + \epsilon_{\rm IJKL} \right] \quad , \\
\varphi^{(2)}{}_{\rm IJKL}(III) &= 4 \left[\delta_{\rm IJ}\delta_{\rm KL} - \delta_{\rm IK}\delta_{\rm JL} + \delta_{\rm IL}\delta_{\rm JK} - \epsilon_{\rm IJKL} \right] \quad , \\
\varphi^{(2)}{}_{\rm IJKL}(V) &= 4 \left[\delta_{\rm IJ}\delta_{\rm KL} - \delta_{\rm IK}\delta_{\rm JL} + \delta_{\rm IL}\delta_{\rm JK} - \epsilon_{\rm IJKL} \right] \quad .
\end{aligned}
\tag{4.10}
$$

One of the striking features of these results is their correlation with an issue about the construction of 4D, $\mathcal{N} = 2$ supermultiplets from 4D, $\mathcal{N} = 1$ supermultiplets. In particular, the pattern of the signs of the coefficients multiplying the ϵ-tensors is quite revealing. The off-shell chiral multiplet sign ($\chi_0(I) = +1$) is opposite to that of the off-shell tensor multiplet ($\chi_0(III) =$ -1) and off-shell vector multiplet ($\chi_0(V) =$ -1) signs.

An off-shell 4D, $\mathcal{N} = 1$ chiral multiplet may be combined with an off-shell 4D, $N = 1$ tensor multiplet to form an off-shell 4D, $\mathcal{N} = 2$ tensor multiplet. An off-shell 4D, $\mathcal{N} = 1$ chiral multiplet may be combined with an off-shell 4D, $N = 1$ vector multiplet to form an off-shell 4D, $\mathcal{N} = 2$ vector multiplet. However, an off-shell 4D, $\mathcal{N} = 1$ tensor multiplet when combined with a 4D, $\mathcal{N} = 1$ vector multiplet forms the so-called 'Vector-Tensor' Multiplet [19]. The Vector-Tensor Multiplet is *not* an off-shell 4D, $\mathcal{N} = 2$ representation. The statement above may be confusing to some of our readers. So let us make clear what we are saying.

The work of [9] implies something that seems to have escaped the general notice of the community familiar with this class of problems. These works in the middle nineties showed that for all values of $\mathcal{N}$, but only in 1D, it is possible to find supermultiplets that have the properties of:

(a.) no off-shell central charges,

(b.) no use of equations of motion, and

(c.) no infinite sets of auxiliary fields.

In other words, the off-shell problem is solved in 1D. Since all the work of the related to the Adinkra/Garden Algebra investigations rests on these fundamental observations, all these studies are within the assumptions (a.), (b.) and (c.) immediately above. Within these restrictions the statement above about the Vector-Tensor Multiplet is correct.

We suspect that this failure on the part of the Vector-Tensor Multiplet to form an off-shell 4D, $\mathcal{N} = 2$ representation is related to the values of χ_0 for the two 4D, $\mathcal{N} = 1$ supermultiplets.

We are led to make some conjectures:

For all off-shell 4D, $\mathcal{N} = 1$ multiplets, the $p = 1$ chromocharacters take the form

$$
\varphi^{(1)}{}_{\rm IJ} = {\rm d}\,\delta_{\rm IJ} \quad , \tag{4.11}
$$

where 2d is the number of bosonic plus fermionic degrees of freedom minus gauge degrees of freedom.

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For all off-shell 4D, $\mathcal{N} = 1$ multiplets, the $p = 2$ chromocharacters take the form

$$\varphi^{(2)}{}_{\rm IJKL} = {\rm d} \left[\delta_{\rm IJ} \delta_{\rm KL} - \delta_{\rm IK} \delta_{\rm JL} + \delta_{\rm IL} \delta_{\rm JK} \right] + \chi_0 \, \epsilon_{\rm IJKL} \quad , \tag{4.12}$$

where χ_0 is a *true* character for classifying the representations of 4D, $\mathcal{N} = 1$ supersymmetry. It is interesting to also note that this character distinguishes between the 2D, $\mathcal{N} = 2$ chiral multiplet versus the twisted chiral multiplet.

Since in the cases of the on-shell chiral multiplet and the on-shell vector multiplet the second line of (3.39) is not satisfied, and also since for the case of the double tensor multiplet neither equation of (3.39) is satisfied, we relegate the calculations of the replacements of these equations to appendix B. Using the results from this appendix we find

$$\begin{aligned}
\varphi^{(2)}{}_{\rm IJKL}(II) &= 2\,\delta_{\rm IJ}\,\delta_{\rm KL} + 2\,[\,\sigma^1 \otimes \sigma^2\,]_{\rm IJ}\,[\,\sigma^1 \otimes \sigma^2\,]_{\rm KL} \quad , \\
\varphi^{(2)}{}_{\rm IJKL}(IV) &= 6\,\delta_{\rm IJ}\,\delta_{\rm KL} + 4\,[\,\sigma^3 \otimes \sigma^3\,]_{\rm IJ}\,[\,\sigma^3 \otimes \sigma^3\,]_{\rm KL} + \\
&\quad + 6\,[\,\sigma^1 \otimes \sigma^2\,]_{\rm IJ}\,[\,\sigma^1 \otimes \sigma^2\,]_{\rm KL} + 4\,[\,\sigma^2 \otimes \sigma^1\,]_{\rm IJ}\,[\,\sigma^2 \otimes \sigma^1\,]_{\rm KL} \\
&\quad + 4\,[\,\sigma^3 \otimes \sigma^1\,]_{\rm IJ}\,[\,\sigma^3 \otimes \sigma^1\,]_{\rm KL} + 4\,[\,\mathbf{I} \otimes \sigma^2\,]_{\rm IJ}\,[\,\mathbf{I} \otimes \sigma^2\,]_{\rm KL} \\
&\quad + 4\,[\,\sigma^1 \otimes \mathbf{I}\,]_{\rm IJ}\,[\,\sigma^1 \otimes \mathbf{I}\,]_{\rm KL} + 4\,[\,\sigma^2 \otimes \sigma^3\,]_{\rm IJ}\,[\,\sigma^2 \otimes \sigma^3\,]_{\rm KL} \\
\varphi^{(2)}{}_{\rm IJKL}(VI) &= 2\,\delta_{\rm IJ}\,\delta_{\rm KL} + 2\,[\,\mathbf{I} \otimes \sigma^2\,]_{\rm IJ}\,[\,\mathbf{I} \otimes \sigma^2\,]_{\rm KL} + 2\,[\,\sigma^2 \otimes \mathbf{I}\,]_{\rm IJ}\,[\,\sigma^2 \otimes \mathbf{I}\,]_{\rm KL} \\
&\quad + 2\,[\,\sigma^2 \otimes \sigma^1\,]_{\rm IJ}\,[\,\sigma^2 \otimes \sigma^1\,]_{\rm KL} \quad .
\end{aligned} \tag{4.13}$$

The forms of the $p = 2$ chromocharacters in the even cases may seem very different from those in the odd cases. But in fact there are similarities.

These similarities become obvious with the use of the generators of the SO(4) rotation group. The six generators of SO(4) can be denoted by $i\,[\alpha^1]_{\rm IJ}$, $i\,[\alpha^2]_{\rm IJ}$, $i\,[\alpha^3]_{\rm IJ}$, $i\,[\beta^1]_{\rm IJ}$, $i\,[\beta^2]_{\rm IJ}$, and $i\,[\beta^3]_{\rm IJ}$ where

$$\begin{aligned}
&[\alpha^1]_{\rm IJ} = [\sigma^2 \otimes \sigma^1]_{\rm IJ} \;, \quad [\alpha^2]_{\rm IJ} = [\mathbf{I} \otimes \sigma^2]_{\rm IJ} \;, \quad [\alpha^3]_{\rm IJ} = [\sigma^2 \otimes \sigma^3]_{\rm IJ} \;, \\
&[\beta^1]_{\rm IJ} = [\sigma^1 \otimes \sigma^2]_{\rm IJ} \;, \quad [\beta^2]_{\rm IJ} = [\sigma^2 \otimes \mathbf{I}]_{\rm IJ} \;, \quad [\beta^3]_{\rm IJ} = [\sigma^3 \otimes \sigma^2]_{\rm IJ} \;,
\end{aligned} \tag{4.14}$$

and these correspond to the fact that locally SO(4) = SU(2) $\times$ SU(2).

In terms of these, the results in (4.10) take the forms

$$\begin{aligned}
\varphi^{(2)}{}_{\rm IJKL}(I) &= 4\left[\delta_{\rm IJ}\delta_{\rm KL} + [\vec{\beta}]_{\rm IJ} \cdot [\vec{\beta}]_{\rm KL}\right] \quad , \\
\varphi^{(2)}{}_{\rm IJKL}(III) &= 4\left[\delta_{\rm IJ}\delta_{\rm KL} + [\vec{\alpha}]_{\rm IJ} \cdot [\vec{\alpha}]_{\rm KL}\right] \quad , \\
\varphi^{(2)}{}_{\rm IJKL}(V) &= 4\left[\delta_{\rm IJ}\delta_{\rm KL} + [\vec{\alpha}]_{\rm IJ} \cdot [\vec{\alpha}]_{\rm KL}\right] \quad ,
\end{aligned} \tag{4.15}$$

and (4.13) becomes

$$\begin{aligned}
\varphi^{(2)}{}_{\rm IJKL}(II) &= 2\,\delta_{\rm IJ}\,\delta_{\rm KL} + 2\,[\,\beta^1\,]_{\rm IJ}\,[\,\beta^1\,]_{\rm KL} \quad , \\
\varphi^{(2)}{}_{\rm IJKL}(IV) &= 6\,\delta_{\rm IJ}\,\delta_{\rm KL} + 6\,[\,\beta^1\,]_{\rm IJ}\,[\,\beta^1\,]_{\rm KL} + 4\,[\,\vec{\alpha}\,\beta^1\,]_{\rm IJ} \cdot [\,\vec{\alpha}\,\beta^1\,]_{\rm KL} + 4\,[\,\vec{\alpha}\,]_{\rm IJ} \cdot [\,\vec{\alpha}\,]_{\rm KL} \\
\varphi^{(2)}{}_{\rm IJKL}(VI) &= 3\,\delta_{\rm IJ}\,\delta_{\rm KL} + 2\,[\,\alpha^2\,]_{\rm IJ}\,[\,\alpha^2\,]_{\rm KL} + 2\,[\,\beta^2\,]_{\rm IJ}\,[\,\beta^2\,]_{\rm KL} + 2\,[\,\alpha^1\,]_{\rm IJ}\,[\,\alpha^1\,]_{\rm KL} \quad .
\end{aligned} \tag{4.16}$$

Thus written, the $p = 2$ chromocharacters for the off-shell theories are seen to have the form of terms dependent on tensor products of the 4×4 identity matrix plus terms that are tensor products in other 4×4 matrices. We see a nice correlation between the spin of the 4D fields and the $p = 2$ chromocharacters. The chiral supermultiplet contained only Lorentz scalars, and the corresponding chromocharacter depends on the β-generators. The vector and tensor supermultiplets contained fields that carried one or more Lorentz vector indices and their chromocharacters depend on the α-generators.

This is the strongest evidence to date that the fourth conjecture made in [10] (though modified now for our change in conventions) is correct and higher dimensional off-shell supersymmetric models can be *faithfully* represented as 1D SUSY models. The spin information of the higher dimensional theory is apparently carried in the chromocharacters associated with the 1D models. Only the off-shell models realize SU(2) symmetries by rotating either the α's among themselves or the β's among themselves. We thus make another conjecture:[3]

> *For all off-shell 4D, $\mathcal{N} = 1$ multiplets, all chromocharacters must possess an SU(2) $\times$ SU(2) symmetry.*

There are strong *purely* algebraic distinctions that must be made between the on-shell and off-shell cases.

In all off-shell representations, the L-matrices and R-matrices are square. This is a consequence of having equal numbers of bosonic and fermionic fields in off-shell supersymmetry representations. The L-matrices and R-matrices satisfy both conditions in (3.39) and that in (3.40). Consequently in off-shell representations, the L-matrices and R-matrices for 1D, $\mathcal{N}$-extended SUSY models are obtained by a projection of $Cl(\mathcal{N} + 1)$. In all off-shell representations each row or column of the L-matrices and R-matrices, when regarded as vectors, form an orthonormal basis set of vectors.

Among the on-shell cases, there is also a strong distinction to be made between the II and VI cases (generic on-shell) and the IV case ("pathogenic" on-shell).

In 'generic' on-shell representations, the L-matrices and R-matrices are *not* square. This is a consequence of having unequal numbers of bosonic and fermionic fields in on-shell supersymmetry representations. The L-matrices and R-matrices satisfy only the first conditions in (3.39) and that in (3.40). In all generic on-shell representations each row or column of the L-matrices and R-matrices, when regarded as vectors, have unit length.

In 'pathogenic' on-shell representations, the L-matrices and R-matrices satisfy only the conditions in (3.40) but not those in (3.39). The L-matrices and R-matrices are generally *not* square. In some 'pathogenic' on-shell representations, each row or column of the L-matrices and R-matrices, when regarded as vectors, do *not* have unit length.

Though we have not discussed them here, there are special pathogenic on-shell representations. Two of the most familiar of these are the 4D, $\mathcal{N} = 2$ Fayet Hypermultiplet [3] and 4D, $\mathcal{N} = 2$ Vector-Tensor Multiplet. Their L-matrices and R-matrices *are* square. However, in these cases, the terms that are the analogs of that given in (2.17) have the

[3] See appendix D for an expanded discussion.

property of being dependent on the equations of motion of the bosonic fields in the multiplet. In this case, these terms are called "off-shell central charges." The initial paper on the Fayet Hypermultiplet introduced such models into the physics literature.

One of our main motivations for including the little known case of the double tensor multiplet was to show that while in 4D theories may superficially appear very similar, after reduction on a 0-brane sharp differences can be seen. It is only in the pathogenic case that the chromocharacters depend on the *products* of α-matrices times β-matrices. This is also a distinction to keep in mind when applying Poincaré duality arguments to supersymmetrical theories. Case VI only differs from case III by the application of a Poincaré duality of one of the spin-0 fields.

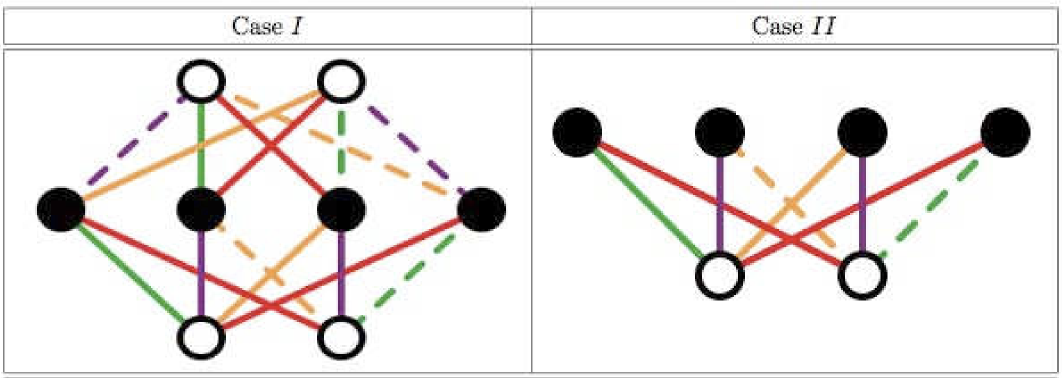
In the next section, we are going to discuss a graphical representation of the results of the current section. This discussion will include all the multiplets seen so far. It should be kept in mind that the double-tensor multiplet has many peculiarities and as no off-shell formulation is known these may not follow the same relations as appear for the other on-shell representations. So many of the comments made about the on-shell multiplets do no apply to the double-tensor multiplet. This should be recalled as the reader goes through the subsequent discussion.

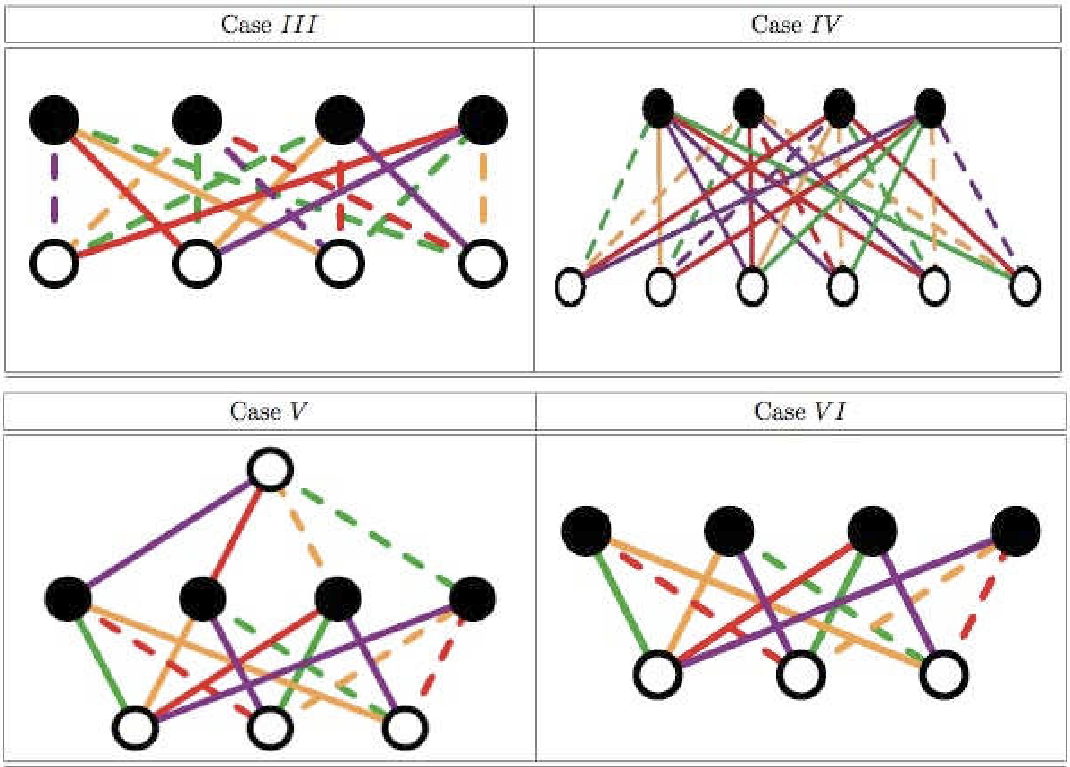
Let us close this section by noting that for the off-shell multiplets, which possess gauge symmetries, the method of 0-brane reduction used has a preferred basis of working in the Coulomb gauge $A_0 = B_{0\,1} = B_{0\,2} = B_{0\,3} = 0$ and this is likely a general feature of this technique.

5 Adinkras from garden algebra matrices

So we have seen from the brief survey of some well-known (and one not well- known) multiplets how the reduction of a supermultiplet on a 0-brane leads to an algebraic association between a given supermultiplet and a set of L-matrices and R-matrices. This was one of the basic observations of [10]. However, *Adinkras* [11] provide a graphical (and vivid) tool that is often convenient as a replacement for the Garden Algebra matrices. We refer the reader to these previous works for detailed explanation of how Adinkras are obtained from reduction on a 0-brane.

We now present the Adinkras for each of the cases I–VI.





(5.1)

It is easily seen that the right hand column entries all contain height-two Adinkras.

This is the case for *all* on-shell theories, as all such theories are valises. The condition of going on-shell corresponds to the erasure of all nodes and links above the second level. The problem of classifying all valise representations is solved and leads to the spectrum of on-shell supersymmetrical theories, a well developed topic in the physics literature.

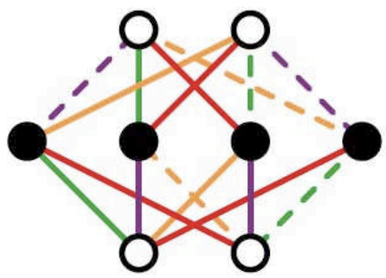
Going beyond on-shell theories and valises requires Adinkras of greater height, as these describe off-shell representations. For a fixed value of $\mathcal{N}$, the maximum height of an Adinkra that realizes $\mathcal{N}$-extended supersymmetry is given by max. height $= \mathcal{N} + 1$. In the discussion of this paper,[4] max. height $= 5$. Using a slight modification of the argument given in [17], it can be proven that no height-5 Adinkra can possess dynamics defined by an action quadratic in the fields of the Adinkra. At height-4, there are known to be two dynamical theories. The most familiar is the complex linear multiplet [20], which will be discussed in a work [21] that is the companion to this paper. Also at height-4, there is the matter gravitino multiplet [22] and some forms of supergravity. The height-3 Adinkras correspond to the familiar off-shell chiral multiplet, the off-shell vector multiplet, and the minimal off-shell supergravity multiplet as the most familiar representatives. The only known off-shell height-2 Adinkra corresponds to the tensor multiplet we have seen in our earlier discussion.

[4] It must be understood that $\mathcal{N}$ refers to the world-line supersymmetries. Thus for four dimensional theories with $\widetilde{\mathcal{N}}$-extended supersymmetry, $\mathcal{N} = 4\,\widetilde{\mathcal{N}}$. A theory with simple supersymmetry in four dimensions requires $\mathcal{N} = 4$ on the world-line.

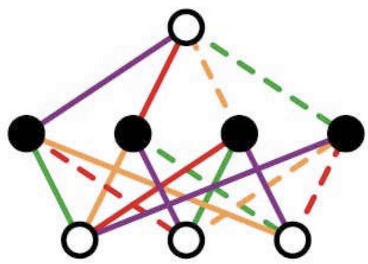
The discussion above also points toward future studies that need to be undertaken to find the Adinkraic representations for all 4D, $\mathcal{N} = 1$ off-shell multiplets. For example, there are many 'variant' representations [23] that are known to exist. The case of supergravity and matter gravitino multiplets will have more information on how higher spin manifests itself at the layers of Adinkras.

5.1 The adinkra transformation group

With Adinkras in hand, there is the possibility to give simplified discussions of some aspects of $\mathcal{GR}$(d, $\mathcal{N}$) formulations. One such issue that is much simplified is that of changing the basis of the representation. Adinkras may be regarded as playing a role similar to Feynman graphs and providing a tool to replace matrix manipulations. To illustrate this, we return to the chiral multiplet and the vector multiplet. From the work of the third section, we have for the chiral multiplet and for the vector multiplet Adinkras given by the following respective images.



$F, G, \psi_1, \psi_2, \psi_3, \psi_4, A, B$ (Order: Left to Right, Top to Bottom)
D_1 - Green, D_2 - Purple, D_3 - Orange, D_4 - Red (5.2)



$d, \lambda_1, \lambda_2, \lambda_3, \lambda_4, A_1, A_2, A_3$ (Order: Left to Right, Top to Bottom)
D_1 - Orange, D_2 - Green, D_3 - Purple, D_4 - Red (5.3)

Using 'root superfields' [10], it is possible to write algebraic expressions for each of these. However, we will eschew such a path and pursue a graphical route to understand the reasons

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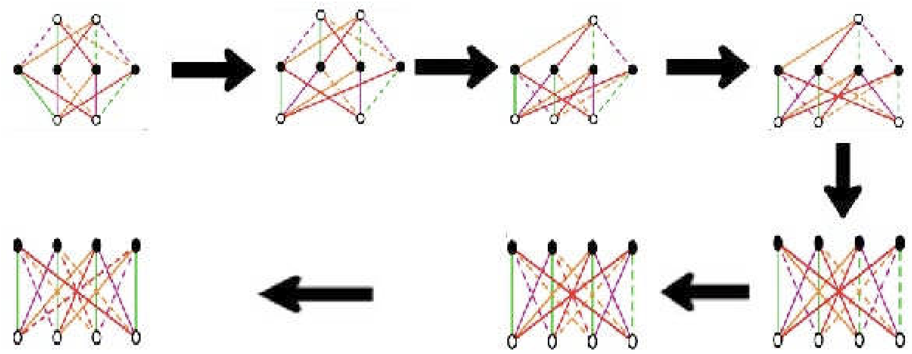
for the different values of χ_0 for the two multiplets. This will provide a graph-theoretical basis for this distinction.

For the work of the DFGHILM collaboration, a graphical piece of software (the Adinkramat — see acknowledgments) was developed for the investigation and manipulation of Adinkras. Using this, one can 'evolve' a given Adinkra into another. The second Adinkra is related to the first by a change of basis and other operations such as 'node raising' and 'node lowering.' Below we will use the Adinkramat to cast the chiral Adinkra into a valise using a maximally symmetric basis. This is shown in the following sequence of operations.[5]



(5.4)

Let us describe the sequence of operations:

(a.) In the first of these, the identity map is applied. Horizontal translations of Adinkra nodes only describe the identity map, *unless* the horizontal ordering of nodes is changed.

(b.) The second operation is a 'node-lowering' one. The exponent of the F-node in the corresponding root superfield is increased by one unit. Also an element of $\mathrm{O}_B(4)$ that exchanges the second and third bosonic nodes was used. Here, the B subscript denotes the $\mathrm{O}(4)$ group that acts on bosonic nodes.

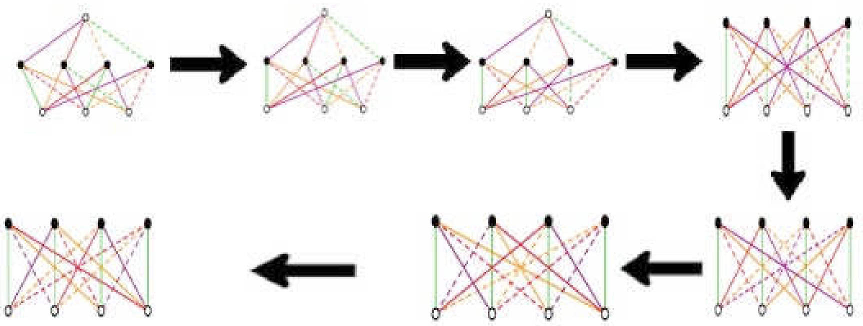
(c.) The third operation is an identity map.

(d.) The fourth operation is a 'node-lowering' one. The exponent of the G-node in the corresponding root superfield is increased by one unit. Also an element of $\mathrm{O}_B(4)$ that exchanges the third and fourth bosonic nodes was used.

(e.) The fifth operation is an element of $\mathrm{O}_B(4)$ that changes the sign of the third and fourth bosonic nodes.

[5] For the benefit of the reader following closely, there is an appendix in which the Adinkra manipulation and standard column and row operations are compared side by side.

A similar sequence of operations may be carried out on the vector multiplet Adinkra using the following sequence of operations.



(5.5)

We again describe the sequence of operations:

(a.) In the first of these, the identity map is applied.

(b.) The second operation is an element of $\mathrm{O}_F(4)$ that exchanges the the second and third fermionic nodes. Here, the F subscript denotes the $\mathrm{O}(4)$ that acts on fermionic nodes.

(c.) The third operation is a 'node-lowering' one. The exponent of the d-node in the corresponding root superfield is increased by one unit.

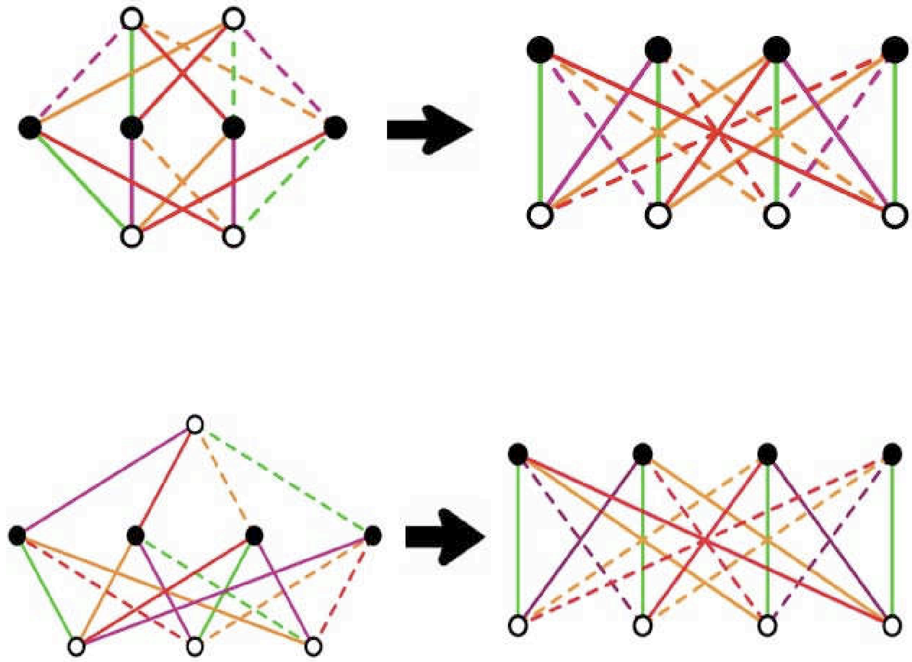
(d.) The fourth operation is an element of $\mathrm{O}_F(4)$ that changes the signs of the third and fourth fermionic nodes.

(e.) In the fifth operation, elements of $\mathrm{O}_B(4)$ and $\mathrm{O}_F(4)$ are used to exchange the location of the first and third bosonic nodes as well as the location of the first and third fermionic nodes. Moreover, some signs were changed.

(f.) The sixth operation is an element of $\mathrm{O}_B(4)$ and $\mathrm{O}_F(4)$ that exchanges the first and second bosonic nodes along with the first and second fermionic nodes.

Thus, under the action of the $\mathrm{O}_B(4)$ and $\mathrm{O}_F(4)$ groups described in (4.3) along with the node-raising and node-lowering group noted for root superfields, we find it possible to implement the following transformations on the Chiral Multiplet and Vector Multiplet

Adinkras.



(5.6)

The Adinkras on the right hand side of (5.6) can be used to ‘read off’[6] the L-matrices associated with each Valise. The L-matrices associated with the uppermost Valise Adinkra are simply

$$(\mathrm{L}_1) = \mathbf{I}_4\,,\ (\mathrm{L}_2) = i\,\sigma^3 \otimes \sigma^2\,,\ (\mathrm{L}_3) = i\,\sigma^2 \otimes \mathbf{I}_2\,,\ (\mathrm{L}_4) = -\,i\,\sigma^1 \otimes \sigma^2\,, \tag{5.7}$$

and L-matrices associated with the lowermost Valise Adinkra are simply

$$(\mathrm{L}_1) = \mathbf{I}_4\,,\ (\mathrm{L}_2) = i\,\sigma^3 \otimes \sigma^2\,,\ (\mathrm{L}_3) = -\,i\,\sigma^2 \otimes \mathbf{I}_2\,,\ (\mathrm{L}_4) = -\,i\,\sigma^1 \otimes \sigma^2\ . \tag{5.8}$$

It can be shown that the chromocharacters associated with these matrices agree with those in (4.11) and (4.15). Also it can be shown there exists a sequence of Adinkra manipulations that take the Tensor Multiplet Adinkra into the lowermost Valise Adinkra.

At first it may seem puzzling that the two Valise Adinkras above can give different chromocharacters. In fact, there is a very small distinction between the two images. It is seen that all the solid orange lines in the first are replaced by dashed orange lines in the second (and vice-versa). This is reflected in the differences in the signs of the L_3 matrices in (5.7) and (5.8). All other colors and dashing match up perfectly. It is apparent that χ_0 is keeping track of this property of the Adinkras! This property of the Adinkra is correlated with 4D fields that carried vector indices versus those without such indices.

It may also seem puzzling that *both* sets of matrices in (5.7) and (5.8) are linearly related to *only* the β-matrices in (4.14). This is due to a very special element that exists among the $\mathcal{X}$ and $\mathcal{Y}$ matrices. It can be shown

$$\Delta\,\alpha_{\mathrm{I}} = \beta_{\mathrm{I}}\,\Delta \quad , \quad (\Delta)^2 = \mathbf{I}_4 \quad , \tag{5.9}$$

[6] See the third appendix also.

where

$$\Delta \ = \ \frac{1}{2} \left[\mathbf{I}_4 \ - \ \vec{\alpha} \cdot \vec{\beta} \right] \quad . \tag{5.10}$$

Thus, by choice of $\mathcal{X}$ and $\mathcal{Y}$, the α's can be 'traded' for the β's and vice-versa.

Finally, the when matrices in (5.7) and (5.8) are written explicitly, it can be shown that they are far more symmetrical than the corresponding matrices in (3.6) and (3.19). In general, were we to randomly lower the upper nodes in the off-shell height-3 Adinkras shown in (5.1), the resultant height-2 Adinkra would appear quite 'cluttered' to the eye. On the other hand, the Adinkras in (5.6) appear quite orderly. It is very satisfying to note that the more symmetrical the matrices, the more symmetrical the Adinkras appear. In fact, the basis used in the Adinkras shown in (5.6) is a maximally symmetrical basis. Calculations are often simpler using such bases.

6 Comparisons to known results

The Adinkra which appears in the upper left of the diagram numbered as equation (5.6) has been named the (2, 4, 2) representation of 1D, $\mathcal{N}$ = 4 supersymmetry (e.g. see the works of [24]). In a similar manner, the Adinkra which appears in the lower left of the diagram numbered as equation (5.6) has been named the (3, 4, 1) representation of 1D, $\mathcal{N}$ = 4 supersymmetry. Adinkras have the property that when 'flipped' about a horizontal axis through the Adinkra there results in a new Adinkra that also describes a supermultiplet. Applying this 'flipping' operation to the (3, 4, 1) representation results in a (1, 4, 3) representations. Although these representations have been given these names in the works of [24], these representations have been known to one of the current authors (SJG) since the presentation of the formula (58) in the work of [10].

The possibility to change the height of nodes (although not expressed using this language) was first discovered in 1994 and shortly thereafter presented in the literature [25]. Later taking advantage of this possibility, the concept of the "root superfield" was introduced [10]. The original definition of this concept[7] was an expression containing exponents whose values determine the height at which nodes appear in a corresponding Adinkra. Our explicit reduction of the component fields of a 4D, $\mathcal{N}$ = 1 chiral multiplet yields a (2, 4, 2) (see also [10]). The reduction of the component fields of a 4D, $\mathcal{N}$ = 1 vector multiplet yields a (3, 4, 1).

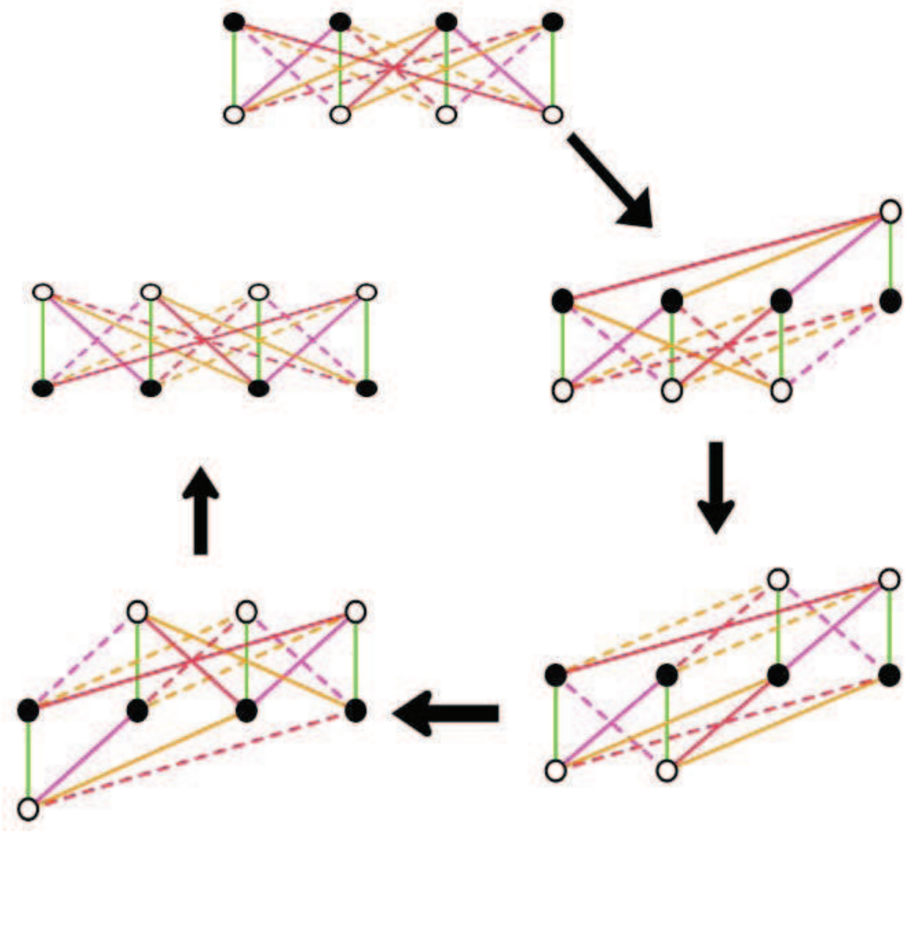
We do not see how the analyses in [24] capture a critical point. If there were a unique 1D, $\mathcal{N}$ = 4 valise (a (4, 4, 0) or root in their conventions), then by raising one node, it could be turned into a (3, 4, 1). Or if two nodes of a unique 1D, $\mathcal{N}$ = 4 valise were raised, it would turn into a (2, 4, 2). Thus, if one made the assumption of a unique 1D, $\mathcal{N}$ = 4 valise, then its two distinct raised-node relatives must be the dimensional reduction of the component fields of a 4D, $\mathcal{N}$ = 1 vector multiplet and the dimensional reduction of the component fields of a 4D, $\mathcal{N}$ = 1 chiral scalar multiplet respectively.

[7] Other authors [26] while retaining the terminology of 'root multiplet', have changed the meaning of the term to only refer to valise Adinkras and associated superfields.

Instead what our calculations show is that the dimensional reduction of the component fields of a 4D, $\mathcal{N} = 1$ chiral scalar multiplet leads to the Adinkra on the upper left hand side of the image numbered as equation (5.6) in this current paper. While the dimensional reduction of the component fields of a 4D, $\mathcal{N} = 1$ vector multiplet leads to the Adinkra on the lower left hand side of the image numbered as equation (5.6) in the current paper.

The Adinkras on the right hand side of (5.6) are distinct, there are no field redefinitions or rearrangements of the bosons among themselves (and the same for the fermions) which will map one of these Adinkras into the other. The degeneracy of the (4,4,0) representation (and corresponding node lifts) is difficult to see in the analyses of [24]. In fact, the distinction between the two valises is reflected in the distinct values found for χ_0 and is exactly the distinction between chiral and twisted chiral multiplets known in 2D, $\mathcal{N} = 2$ theories. This result had been *surmised* in other work by the DFGHILM collaboration. The calculations in this paper are the first to *prove* this is the case and shows the value of why explicit calculations need to be performed to support the many conjectures made solely by looking at the 1D structure of these theories.

One other matter we will attempt to make clear for our readers what is the meaning of root superfields, *as originally defined in* [10] and how are these related to higher 4D, $\mathcal{N} = 1$ representations. The original meaning of a root multiplet or root superfield is that this term refers to set of distinct ordinary superfields that form part of a web obtained by raising and lowering nodes. Thus the complete root superfield associated with the upper part of the diagram in equation (5.6) takes the form

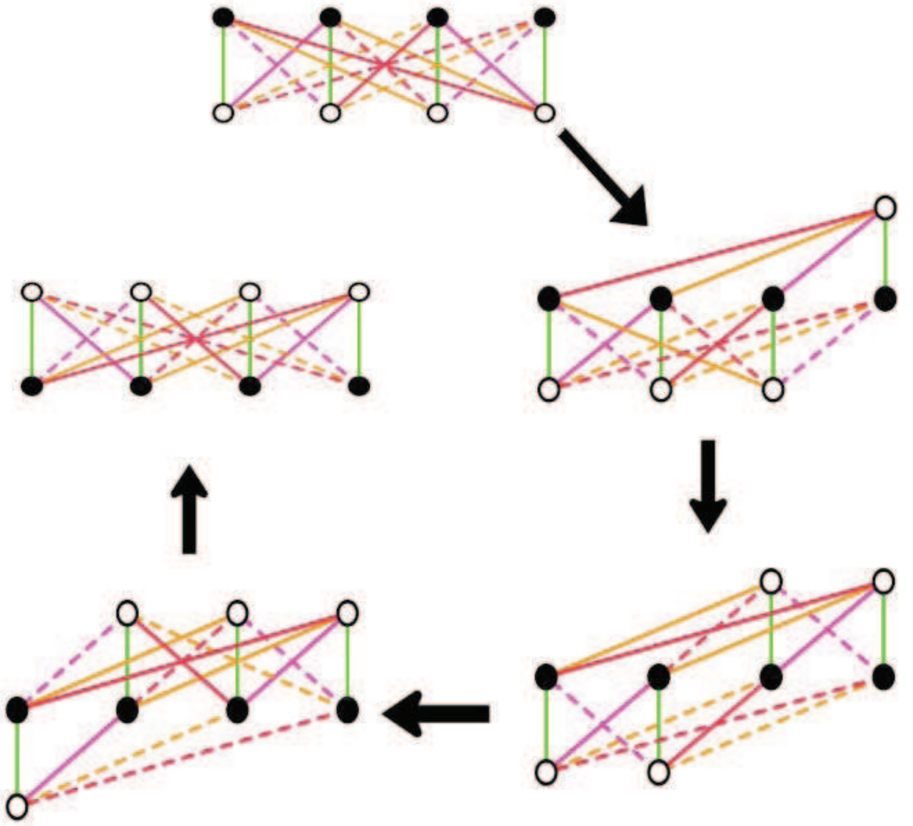


(6.1)

and this diagram explicitly shows the transformations

$$(4,\ 4,\ 0)_c \ \rightarrow\ (3,\ 4,\ 1)_c \ \rightarrow\ (2,\ 4,\ 2)_c \ \rightarrow\ (1,\ 4,\ 3)_c \ \rightarrow\ (0,\ 4,\ 4)_c \quad . \tag{6.2}$$

In a similar manner the complete root superfield associated with the lower part of the diagram in equation (5.6) takes the form



(6.3)

and this diagram explicitly shows the transformations

$$(4,\ 4,\ 0)_t \ \rightarrow\ (3,\ 4,\ 1)_t \ \rightarrow\ (2,\ 4,\ 2)_t \ \rightarrow\ (1,\ 4,\ 3)_t \ \rightarrow\ (0,\ 4,\ 4)_t \quad . \tag{6.4}$$

The works of [27] show that there is an exclusion principle-like nature to lifting these multiplets to 4D. One can only 'oxidize' the $(2,\ 4,\ 2)_c$ to become a 4D, $\mathcal{N} = 1$ chiral scalar multiplet and one can only 'oxidize' the $(3,\ 4,\ 1)_t$ to become a 4D, $\mathcal{N} = 1$ vector multiplet. This sort of behavior is what was anticipated in [11]. Only a very limited number of representations in the lower dimension can be oxidized among members of a root superfield. The only ambiguity found is one that amounts to a re-definition of the relation of which of two right hand Adinkras in (5.6) is chosen as a starting point.

7 Conclusion

In this present work, there has been presented a survey of features that occur in the study of embedding 4D, $\mathcal{N} = 1$ supersymmetrical systems into the context of Adinkras and Garden Algebras. We have explicitly demonstrated that off-shell supersymmetrical multiplets lead, upon reduction on 0-branes, to a universal algebraic structure described by (3.39) and (3.40) that we refer to as defining a mathematical structure denoted by $\mathcal{GR}(\mathrm{d}, \mathcal{N})$. On the other hand, we have shown that on-shell theories typically lead to an algebraic characterization in terms of $\mathcal{GR}(\mathrm{d}_L, \mathrm{d}_R, \mathcal{N})$.

The structures we have discussed allow for a completely algebraic characterization of “The Fundamental Supersymmetry Challenge” (see final work in [9]). The 0-brane reduction of all supersymmetrical theories (including all ten and eleven dimensional ones) is conceptually no different from the exercises undertaken in the third section for the on-shell chiral multiplet (equations (2.4)–(2.6) and (3.11)–(3.13)) and vector multiplets (equations (2.21)–(2.23) and (3.37)–(3.38)). Thus, ten and eleven dimensional on-shell supersymmetrical multiplets possess derivable $\mathcal{GR}(\mathrm{d}_L, \mathrm{d}_R, \mathcal{N})$ representations in terms of L-matrices and R-matrices similar to those in (3.12), (3.13), and (3.38). In the case of the on-shell chiral and vector multiplets, their L-matrices and R-matrices ((3.12), (3.13), and (3.38)) can be embedded into the L-matrices and R-matrices ((3.6), (3.9), and (3.34)) of the off-shell chiral and vector multiplets.

We can thus state the first part of the fundamental supersymmetry challenge solely as a algebraic problem: ‘When can a given representation of $\mathcal{GR}(\mathrm{d}_L, \mathrm{d}_R, \mathcal{N})$ be embedded into $\mathcal{GR}(\mathrm{d}, \mathcal{N})$? The answer to this question may hold a key to obtaining some interesting results.

With regard to $\mathcal{GR}(\mathrm{d}, \mathcal{N})$ versus $\mathcal{GR}(\mathrm{d}_L, \mathrm{d}_R, \mathcal{N})$, we have been able to advance the state-of-the-art understanding. From the part of our survey comparing off-shell versus on-shell multiplets, we have found that when viewed from the perspective of one dimension, the main difference between them lies in regard to a chiral $\mathrm{SU}(2) \times \mathrm{SU}(2)$ theory. Off-shell theories possess full invariance with regard to all of the chiral $\mathrm{SU}(2) \times \mathrm{SU}(2)$ group, while on-shell theories possess symmetry only with respect to a broken sub-group.

The two Adinkras in (5.6) show a remarkable resemblance to the cis-trans isomerism well known in chemistry. Specifically in fact, we can refer to the uppermost Adinkra as the 4D, $\mathcal{N} = 1$ cis-Valise[8] and the second Adinkra as the 4D, $\mathcal{N} = 1$ trans-Valise. We believe these are to 4D, $\mathcal{N} = 1$ representation theory as quarks and anti-quarks are to SU(3).

However, we know from the current understanding of the work of the DFGHILM collaboration, that the analogs of higher $\mathcal{N}$ studies show an incredible proliferation of representations that valise Adinkras produce. This rich spectrum of representations is more reminiscent of biology and genetics instead of the representation theory normally seen in physics. Because of this, we have been influenced in our studies by genomics in particular. From this vantage point, it would perhaps be appropriate to refer to the cis-Valise and trans-Valise as ‘genes.’

This leads us to a final conjectures:

The cis-Valise and trans-Valise are the fundamental 4D, $\mathcal{N} = 1$ genes from which all off-shell 4D, $\mathcal{N} = 1$ supersymmetry representations can be derived.

[8] In recognition that the sign of the ϵ-term is the same as the first term in the expression this is appropriate.

Should this conjecture be true, it implies that for the genetic classification of all 4D, $\mathcal{N} = 1$ supersymmetry representations, at least the two integers n_t and n_c (which give the number of trans-Valises and cis-Valises contained in a general representation) are required.

In a number of presentations by one of the authors (SJG), the expression, 'the DNA of Reality,' has been used. Our current work provides the most detailed explanation to date for why this may be more than merely metaphorical.

> *"My methods are really methods of working and thinking; this is why they have crept in everywhere anonymously."* - Emmy Noether

Note Added. After the conclusion of this work, two papers have appeared on the arXiv which provide some specific examples of how Adinkras via the Garden Algebras provide a 1D holographic description of 4D, $\mathcal{N} = 1$ supermultiplets. These works can be found in papers cited as [27].

Acknowledgments

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A Conventions for gamma matrices

Our conventions for the four dimensional discussion are such that we use real four component spinors (when their indices are in an up position). Our choice of Minkowski metric is the 'mostly plus metric.'

We use the outer product to write our 4 x 4 matrices in terms of 2 x 2 matrices. If M and N are two such matrices where

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad , \quad N = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \tag{A.1}$$

then we choose our conventions so that

$$M \otimes N = \begin{pmatrix} m_{11}\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} & m_{12}\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \\ m_{21}\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} & m_{22}\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} m_{11}n_{11} & m_{11}n_{12} & m_{12}n_{11} & m_{12}n_{12} \\ m_{11}n_{21} & m_{11}n_{22} & m_{12}n_{21} & m_{12}n_{22} \\ m_{21}n_{11} & m_{21}n_{12} & m_{22}n_{11} & m_{22}n_{12} \\ m_{21}n_{21} & m_{21}n_{22} & m_{22}n_{21} & m_{22}n_{22} \end{pmatrix} \quad . \tag{A.2}$$

In this notation, the four dimensional gamma matrices we use are defined by

$$\begin{aligned} (\gamma^0)_a{}^b &= i(\sigma^3 \otimes \sigma^2)_a{}^b \quad , \qquad (\gamma^1)_a{}^b = (\mathbf{I}_2 \otimes \sigma^1)_a{}^b \quad , \\ (\gamma^2)_a{}^b &= (\sigma^2 \otimes \sigma^2)_a{}^b \quad , \qquad (\gamma^3)_a{}^b = (\mathbf{I}_2 \otimes \sigma^3)_a{}^b \quad . \end{aligned} \tag{A.3}$$

which can all be seen to be purely imaginary. The corresponding gamma-5 matrix is given by

$$(\gamma^5)_a{}^b = -(\sigma^1 \otimes \sigma^2)_a{}^b \quad . \tag{A.4}$$

Some useful Identities then follow

$$\begin{aligned} \gamma^\mu\,\gamma^\nu \;+\; \gamma^\nu\,\gamma^\mu \;&=\; 2\,\eta^{\mu\,\nu}\,\mathbf{I}_4 \;, \qquad \gamma^\mu\,\gamma_\mu \;=\; 4\,\mathbf{I}_4 \;, \qquad \gamma^\mu\,\gamma_\alpha\,\gamma_\mu \;=\; -\,2\,\gamma_\alpha \quad , \\ \gamma^5\,[\,\gamma^\alpha\,,\;\gamma^\beta\,] \;&=\; -\,i\frac{1}{2}\,\epsilon^{\alpha\,\beta\,\mu\,\nu}\,[\,\gamma_\mu\,,\;\gamma_\nu\,] \quad , \qquad \gamma^\mu\,[\,\gamma^\alpha\,,\;\gamma^\beta\,]\,\gamma_\mu \;=\; 0 \quad , \\ \gamma^\mu\,[\,\gamma^\alpha\,,\;\gamma^\beta\,] \;&=\; 2\,[\;\eta^{\mu\,\alpha}\,\gamma^\beta \;-\; \eta^{\mu\,\beta}\,\gamma^\alpha\;] \;+\; i\,2\,\epsilon^{\alpha\,\beta\mu\,\nu}\gamma^5\gamma_\nu \quad , \\ [\,\gamma^\alpha\,,\;\gamma^\beta\,]\,\gamma^\mu \;&=\; -\,2\,[\;\eta^{\mu\,\alpha}\,\gamma^\beta \;-\; \eta^{\mu\,\beta}\,\gamma^\alpha\;] \;+\; i\,2\,\epsilon^{\alpha\,\beta\mu\,\nu}\gamma^5\gamma_\nu \quad . \end{aligned} \tag{A.5}$$

In order to raise and lower spinor indices, we define a spinor metric by

$$C_{ab} \equiv -i(\sigma^3 \otimes \sigma^2)_{ab} \;=\; \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \rightarrow \quad C_{ab} = -C_{ba} \quad . \tag{A.6}$$

The inverse spinor metric is defined by the condition $C^{ab}C_{ac} = \delta_c{}^b$.

The second rank anti-symmetric matrix is defined by

$$(\sigma^{\mu\nu})_a{}^b \;\equiv\; \frac{i}{2}[(\gamma^\mu)_a{}^c(\gamma^\nu)_c{}^b \;-\; (\gamma^\nu)_a{}^c(\gamma^\mu)_c{}^b] \quad . \tag{A.7}$$

Next a direct set of calculations show the following properties:

$$(\gamma^\mu)_a{}^c C_{cb} \;=\; (\gamma^\mu)_b{}^c C_{ca} \quad . \tag{A.8}$$

$$(\sigma^{\mu\nu})_{ab} \;=\; (\sigma^{\mu\nu})_{ba} \quad , \tag{A.9}$$

$$(\gamma^5\gamma^0)_a{}^b \;=\; -(\sigma^2 \otimes \mathbf{I}_2)_a{}^b \quad ,$$

$$
\begin{aligned}
(\gamma^5\gamma^1)_a{}^b &= i(\sigma^1\otimes\sigma^3)_a{}^b \quad , \\
(\gamma^5\gamma^2)_a{}^b &= -i(\sigma^3\otimes\mathbf{I}_2)_a{}^b \quad , \\
(\gamma^5\gamma^3)_a{}^b &= -i(\sigma^1\otimes\sigma^1)_a{}^b \quad ,
\end{aligned} \tag{A.10}
$$

$$
(\gamma^5\gamma^\mu)_a{}^c C_{cb} = -(\gamma^5\gamma^\mu)_b{}^c C_{ca} \tag{A.11}
$$

B $\mathcal{GR}(\mathrm{d}_L, \mathrm{d}_R, \mathcal{N})$ closure terms

In the case of the on-shell Chiral Multiplet we find

$$
(\,\mathrm{R}_{\mathrm{J}}\,)_{\hat{i}}{}^{j}\,(\,\mathrm{L}_{\mathrm{I}}\,)_{j}{}^{\hat{k}} + (\,\mathrm{R}_{\mathrm{I}}\,)_{\hat{i}}{}^{j}\,(\,\mathrm{L}_{\mathrm{J}}\,)_{j}{}^{\hat{k}} = \;\delta_{\mathrm{IJ}}\,(\mathbf{I})_{\hat{i}}{}^{\hat{k}} \;+\; [\,\vec{\alpha}\beta^1\,]_{\mathrm{IJ}}\cdot(\,\vec{\alpha}\beta^1\,)_{\hat{i}}{}^{\hat{k}} \quad . \tag{B.1}
$$

in place of the second equation of (3.39).

In the case of the Double Tensor Multiplet we will calculate the left hand sides of both the first and second equations in (3.39). We find

$$
(\,\mathrm{L}_{\mathrm{I}}\,)_{i}{}^{\hat{j}}\,(\,\mathrm{R}_{\mathrm{J}}\,)_{\hat{j}}{}^{k} + (\,\mathrm{L}_{\mathrm{J}}\,)_{i}{}^{\hat{j}}\,(\,\mathrm{R}_{\mathrm{I}}\,)_{\hat{j}}{}^{k} = \;2\,\delta_{\mathrm{IJ}}\,(\,\mathbf{I}_2\otimes\mathbf{I}_3\,)_{i}{}^{k} \;-\; 2\,[\,\vec{\alpha}\,\beta^1\,]_{\mathrm{IJ}}\cdot(\,\sigma^2\otimes\vec{J}\,)_{i}{}^{k} \quad , \tag{B.2}
$$

where in writing this expression, we have introduced the dimensionless generators of spin-1 angular momentum denoted by J_1, J_2 and J_3. We simply note

$$
\begin{aligned}
J_1 \;&=\; \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -\,i & 0 & 0 \end{pmatrix} \;,\;\; J_2 \;=\; \begin{pmatrix} 0 & i & 0 \\ -\,i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;, \\
J_3 \;&=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -\,i & 0 \end{pmatrix} \quad .
\end{aligned} \tag{B.3}
$$

satisfy the commutation relationships

$$
[\,J_i\,,\;J_j\,] \;=\; i\,\epsilon_{i\,j\,k}J_k \quad . \tag{B.4}
$$

These relations are recognized as those for the usual generators of angular momentum. We can continue and find the result

$$
(\,\mathrm{R}_{\mathrm{J}}\,)_{\hat{i}}{}^{j}\,(\,\mathrm{L}_{\mathrm{I}}\,)_{j}{}^{\hat{k}} + (\,\mathrm{R}_{\mathrm{I}}\,)_{\hat{i}}{}^{j}\,(\,\mathrm{L}_{\mathrm{J}}\,)_{j}{}^{\hat{k}} \;=\; 3\,\delta_{\mathrm{IJ}}\,(\mathbf{I})_{\hat{i}}{}^{\hat{k}} \;-\; [\,\vec{\alpha}\beta^1\,]_{\mathrm{IJ}}\cdot(\,\vec{\alpha}\beta^1\,)_{\hat{i}}{}^{\hat{k}} \;, \tag{B.5}
$$

which is very similar to the case of the on-shell chiral multiplet given above (B.1). This similarity is so striking that one might hope for its universality. All such hopes vanish from the same calculation in the context of the on-shell Vector Multiplet.

For the second equation in (3.39) evaluated on the Vector Multiplets we find the result

$$
\begin{aligned}
(\,\mathrm{R}_{\mathrm{J}}\,)_{\hat{i}}{}^{j}\,(\,\mathrm{L}_{\mathrm{I}}\,)_{j}{}^{\hat{k}} + (\,\mathrm{R}_{\mathrm{I}}\,)_{\hat{i}}{}^{j}\,(\,\mathrm{L}_{\mathrm{J}}\,)_{j}{}^{\hat{k}} \;=\; & \frac{3}{2}\,\delta_{\mathrm{IJ}}\,(\,\mathbf{I}_4\,)_{\hat{i}}{}^{\hat{k}} \;-\; \frac{1}{2}\,[\,\vec{\alpha}\,\beta^2\,]_{\mathrm{IJ}}\cdot(\,\vec{\alpha}\,\beta^2\,)_{\hat{i}}{}^{\hat{k}} \\
& +\frac{1}{2}\,[\,\vec{\alpha}\,\beta^1\,]_{\mathrm{IJ}}\cdot(\,\vec{\alpha}\,\beta^1\,)_{\hat{i}}{}^{\hat{k}} \\
& +\frac{1}{2}\,[\,\vec{\alpha}\,\beta^3\,]_{\mathrm{IJ}}\cdot(\,\vec{\alpha}\,\beta^3\,)_{\hat{i}}{}^{\hat{k}} \quad .
\end{aligned} \tag{B.6}
$$

Interestingly, these calculations show some general regularities though still to a large degree, the exact nature of $\mathcal{GR}(\mathrm{d}_L, \mathrm{d}_R, \mathcal{N})$ for $\mathrm{d}_L \neq \mathrm{d}_R$ remains a mystery.

Aside from the identity matrix common to (B.1), (B.2), (B.5), and (B.6), there is an interesting similarity of the matrices that do appear on the right sides of the equations. These matrices that appear in (B.1), (B.2), (B.5), and (B.6), can be expressed in the form utilizing matrix representations of SU(2) and are characteristic of theories realizing broken chiral SU(2) $\times$ SU(2) symmetries and none of the results in this appendix respect the full chiral SU(2) $\times$ SU(2) symmetry group seen in the off-shell theories.

C A primer on adinkra transformations

In the discussion of section five, the two Adinkras shown in (5.6) were used to generate the corresponding matrices in (5.7) and (5.8) without explanation of the intervening steps. In an effort to be as transparent as possible, in this short appendix we present an explanation on how to read an Adinkra and generate the corresponding matrices.

A large class of the solutions to the conditions in (3.39) and (3.40) have the property that L-matrices and R-matrices contain rows and columns with:

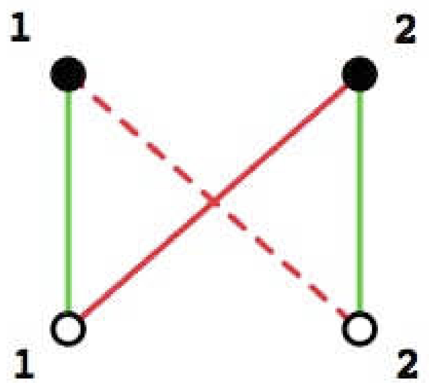
(a.) each row (when regarded as a vector) is a unit vector,

(b.) each column (when regarded as a vector) is a unit vector, and

(c.) the set of d row-vectors (or column-vectors) is an orthonormal set.

Taken together, these conditions imply the entries in these matrices are equal to +1, 0, or -1. Our conventions are such that we use solid lines to indicate a value of + 1, a dashed line to indicate a value of -1 and no line at all to indicate a zero entry. The conditions in (5.7) and (5.7) require $\mathcal{N}$ linearly independent matrices in order for the representation to be faithful. For this purpose, $\mathcal{N}$ distinct colors are used in an Adinkra.

Rather than continue with a recitation of rules, it is easier to begin with a simple example. The basic $\mathcal{N} = 2$ Adinkra appears as below.

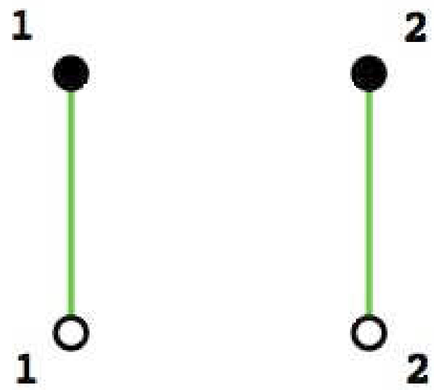


(C.1)

For the purpose of this appendix, we have numbered the white nodes and the black nodes.

Instead of regarding the white nodes as bosons and the black nodes as fermions, we can instead think of the white nodes as being associated with the rows in a matrix and the black nodes as being associated with the columns in a matrix. This is a two-color Adinkra. So it is necessarily associated with two matrices that we can denote by L_1 and L_2. We have a choice on how to associate which matrix with which color so we choose to associate the green edges with L_1 and the red edges with L_2.

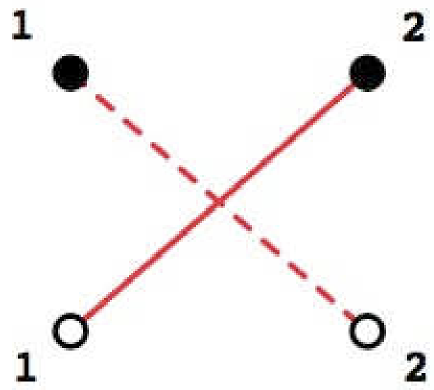
In order to concentrate on L_1, the Adinkra (C.1) may be viewed through a "green-pass" filter that only allows the green edges to show. Thus we arrive at the image below.



(C.2)

The information contained in this image is a factor of 1 appears in the first row and first column of the matrix as well as a factor of 1 appears in the second row and second column of the matrix. In other words this is the identity matrix $\mathbf{I}_2$.

In order to concentrate on L_2, the Adinkra (C.1) may be viewed through a "red-pass" filter that only allows the red edges to show. Thus we arrive at the image below.



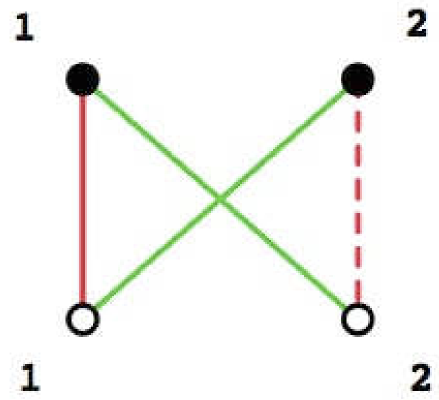
(C.3)

The information contained in this image is that a factor of 1 appears in the first row and second column of the matrix as well as a factor of $-\,1$ appears in the second row and first column of the matrix. In other words this is the matrix $i\,\sigma^2$. So the Adinkra in (C.1) is associated with L_1 and L_2 via the equation

$$(\,\mathrm{L}_1,\,\mathrm{L}_2\,) \;=\; \big(\mathbf{I}_2,\,i\sigma^2\big) \quad . \tag{C.4}$$

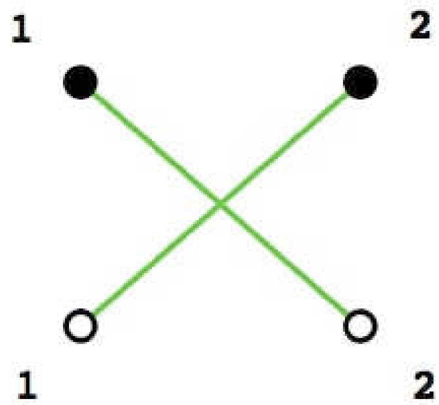
It is notable that with complete fidelity, all the information of the matrices are contained in the Adinkra. In other words the Adinkra is a faithful representation of these matrices.

In a similar manner, the Adinkra whose image appears immediately below



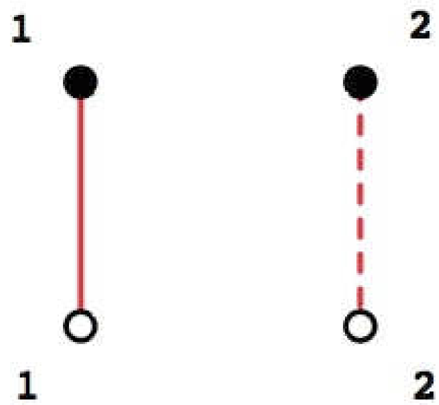
(C.5)

possesses a 'green-pass' filtered image of the form



(C.6)

and possesses a 'red-pass' filtered image of the form below.



(C.7)

Clearly, the Adinkra in (C.1) is different from the one in (C.5). So the matrices associated with the latter cannot be the same as those associated with the former. We will denote the matrices associated with the latter by $\widehat{\mathrm{L}}_1$ and $\widehat{\mathrm{L}}_2$. By using the same logic that led to (C.4) we find

$$\left(\widehat{\mathrm{L}}_1,\, \widehat{\mathrm{L}}_2\right) \;=\; \left(\sigma^1,\, \sigma^3\right) \quad . \tag{C.8}$$

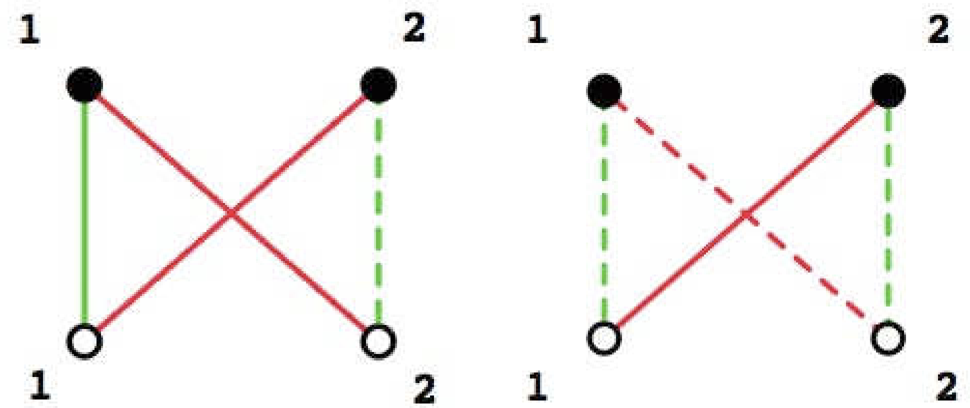
However, there is a visual relation between (C.1) and (C.5). If the two black nodes at the top of the first Adinkra are exchanged, then Adinkra (C.1) changes into Adinkra (C.5). Furthermore, it can be verified that the sets of matrices given in (C.4) and (C.8) satisfy the conditions in (4.1) and (4.2).

In equations (4.3) and (4.4) there were defined matrices $\mathcal{X}$ and $\mathcal{Y}$ that transform L-matrices and R-matrices along orbits and define a class structure. It might be possible to work out the explicit forms of $\mathcal{X}$ and $\mathcal{Y}$ to relate the matrices in (C.4) to those in (C.8). It is straightforward calculation to show the required matrices take the forms

$$\mathcal{X} \;=\; \frac{k_1\,\mathbf{I} \;+\; i\,k_2\,\sigma^2}{\sqrt{\,k_1^2 \;+\; k_2^2\,}} \quad , \quad \mathcal{Y} \;=\; \frac{k_1\,\sigma^1 \;-\; k_2\,\sigma^3}{\sqrt{\,k_1^2 \;+\; k_2^2\,}} \quad , \tag{C.9}$$

where k_1 and k_2 are arbitrary real parameters. Notice for the choice $k_2 = 0$, this set of transformation corresponds to the identity map acting on the white nodes and a pure exchange on the black nodes as was the visual intuition gained by comparing (C.1) to (C.5). The two matrices in (C.9) effectuate the exchange of the two closed nodes that occur in the transformation from (C.1) to (C.5).

Two additional $\mathcal{N} = 2$ Adinkras are shown in (C.10).



(C.10)

For the leftmost image, we have

$$\left(\widetilde{\mathrm{L}}_1,\, \widetilde{\mathrm{L}}_2\right) \;=\; \left(\sigma^3,\, \sigma^1\right) \quad , \tag{C.11}$$

and for the rightmost image, there is

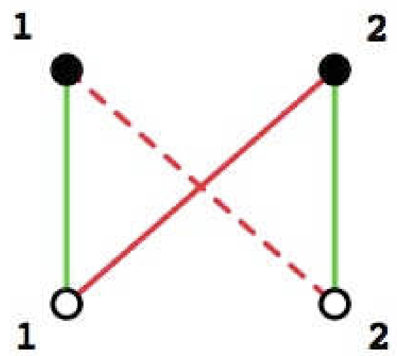
$$\left(\overline{\mathrm{L}}_1,\, \overline{\mathrm{L}}_2\right) \;=\; \left(-\,\mathbf{I}_2,\, i\,\sigma^2\right) \quad . \tag{C.12}$$

One choice of $\mathcal{X}$ and $\mathcal{Y}$ which relates the first of these to (C.1) is given by $\mathcal{X} = \sigma^3$ and $\mathcal{Y} = \mathbf{I}_2$. This effectuates a change of sign to the links attached to the open node at position 2 in the image of (C.1). For the second in (C.10) one set of matrices we see $\mathcal{X} = -\,\sigma^3$ and

$\mathcal{Y} = \sigma^3$ will relate it to (C.1). This effectuates a change of sign to the links attached to the open node at position 1 *and* a change of sign to the links attached to the open closed node at position 2.

With a bit of practice, it become very simple to use an Adinkra to generate a corresponding set of matrices. However, the real advantage of Adinkras, used in the work of the DFGHILM collaboration, is the ability to visually manipulate (using the Adinkramat) these images to change basis and generally investigate the Garden Algebra matrices.

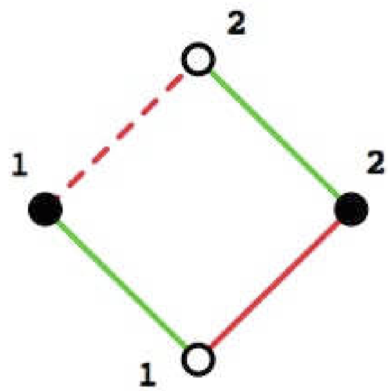
The Adinkra of (C.1) is also associated with a collection of superfields and spinorial differential equations that relate them.



$$\begin{aligned}
D_1\Phi_1 &= i\,\Psi_1\,, & D_2\Phi_1 &= i\,\Psi_2\,, \\
D_1\Phi_2 &= i\,\Psi_2\,, & D_2\Phi_2 &= -\,i\,\Psi_1\,, \\
D_1\Psi_1 &= \partial_\tau\Phi_1\,, & D_2\Psi_1 &= -\,\partial_\tau\Phi_2\,, \\
D_1\Psi_2 &= \partial_\tau\Phi_2\,, & D_2\Psi_2 &= \partial_\tau\Phi_1\,.
\end{aligned} \tag{C.13}$$

Here the bosonic superfields Φ_1 and Φ_2 are associated with the open nodes # 1 and # 2 at the lowest level of the Adinkra. The fermionic superfields Ψ_1 and Ψ_2 are associated with the closed nodes # 1 and # 2 at the highest level of the Adinkra. The spinorial derivative D_1 is associated with green edges and the spinorial derivative D_2 is associated with red edges.

The process of "lifting a node" can be shown by first making one local redefinition and one the non-local redefinition; $\Phi_1 \rightarrow A$, $\Phi_2 \rightarrow \partial_\tau^{-1}F$. Due to the second equation, the field F has a higher engineering dimension than Φ_2 and accordingly the node associated with it is lifted in the Adinkra which can be redrawn as



$$\begin{aligned}
D_1 A &= i\,\Psi_1\,, & D_2 A &= i\,\Psi_2\,, \\
D_1 F &= i\,\partial_\tau\Psi_2\,, & D_2 F &= -\,i\,\partial_\tau\Psi_1\,, \\
D_1\Psi_1 &= \partial_\tau A\,, & D_2\Psi_1 &= -\,F\,, \\
D_1\Psi_2 &= F\,, & D_2\Psi_2 &= \partial_\tau A\,.
\end{aligned} \tag{C.14}$$

The association of one superfield with each node in an Adinkra and the association of each Adinkra color-edge with a distinct D-operator was implicitly introduced in the work of the DFGHILM collaboration seen in [17].

D SU(2) $\otimes$ SU(2) symmetry and quartic chromocharacters

For convenience of this discussion, let us begin by gathering the quartic chromocharacters here below

$$\begin{aligned}
\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(I) &= 4\left[\delta_{\rm I}{}^{\rm J}\delta_{\rm K}{}^{\rm L} + [\vec{\beta}]_{\rm I}{}^{\rm J}\cdot[\vec{\beta}]_{\rm K}{}^{\rm L}\right] \quad , \\
\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(III) &= 4[\ \delta_{\rm I}{}^{\rm J}\delta_{\rm K}{}^{\rm L} + [\vec{\alpha}]_{\rm I}{}^{\rm J}\cdot[\vec{\alpha}]_{\rm K}{}^{\rm L}\ ] \quad , \\
\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(V) &= 4[\ \delta_{\rm I}{}^{\rm J}\delta_{\rm K}{}^{\rm L} + [\vec{\alpha}]_{\rm I}{}^{\rm J}\cdot[\vec{\alpha}]_{\rm K}{}^{\rm L}\ ] \quad , \\
\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(II) &= 2\delta_{\rm I}{}^{\rm J}\delta_{\rm K}{}^{\rm L} + \ 2[\beta^1]_{\rm I}{}^{\rm J}[\beta^1]_{\rm K}{}^{\rm L} \quad , \\
\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(IV) &= 6\delta_{\rm I}{}^{\rm J}\delta_{\rm K}{}^{\rm L} + 6[\beta^1]_{\rm I}{}^{\rm J}[\beta^1]_{\rm K}{}^{\rm L} + 4[\vec{\alpha}\beta^1]_{\rm I}{}^{\rm J}\cdot[\vec{\alpha}\beta^1]_{\rm K}{}^{\rm L} + 4[\vec{\alpha}]_{\rm I}{}^{\rm J}\cdot[\vec{\alpha}]_{\rm K}{}^{\rm L} \\
\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(VI) &= 3\delta_{\rm I}{}^{\rm J}\delta_{\rm K}{}^{\rm L} + 2[\alpha^2]_{\rm I}{}^{\rm J}[\alpha^2]_{\rm K}{}^{\rm L} + 2[\beta^2]_{\rm I}{}^{\rm J}[\beta^2]_{\rm K}{}^{\rm L} + \ 2[\alpha^1]_{\rm I}{}^{\rm J}[\alpha^1]_{\rm K}{}^{\rm L} \quad , \qquad \text{(D.1)}
\end{aligned}$$

where we have used a Euclidean metric to raise a pair of indices. We next observe the relations

$$\begin{aligned}
\left[\alpha^{\rm A}\,,\ \alpha^{\rm B}\right] &= i\,2\,\epsilon^{\rm A\,B\,C}\alpha^{\rm C}\,, \qquad\qquad \left[\beta^{\rm A}\,,\ \beta^{\rm B}\right] = i\,2\,\epsilon^{\rm A\,B\,C}\beta^{\rm C} \quad , \\
\left[\alpha^{\rm A}\,,\ \alpha^{\rm B}\right] &= 0 \quad . \qquad\qquad \text{(D.2)}
\end{aligned}$$

These imply that a group element of SU(2) $\otimes$ SU(2) denoted by $\mathcal{G}$ can be written in the form

$$[\mathcal{G}(u,v)]_{\rm I}{}^{\rm J} \ = \ \left[exp\left(i\frac{1}{2}\,u^{\rm A}\alpha^{\rm A}\right)exp\left(i\frac{1}{2}\,v^{\rm A}\beta^{\rm A}\right)\right]_{\rm I}{}^{\rm J} \quad . \qquad \text{(D.3)}$$

We next calculate the following results

$$\begin{aligned}
\left[\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(I)\right]' &= [\mathcal{G}(u,v)]_{\rm I}{}^{\rm R}\,[\mathcal{G}(u,v)]_{\rm J}{}^{\rm T}\varphi^{(2)}{}_{\rm R}{}^{\rm S}{}_{\rm T}{}^{\rm U}(I)\,\left[\mathcal{G}^{-1}(u,v)\right]_{\rm S}{}^{\rm J}\,\left[\mathcal{G}^{-1}(u,v)\right]_{\rm U}{}^{\rm L} \\
&= \varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(I) \quad , \\
\left[\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(III)\right]' &= [\mathcal{G}(u,v)]_{\rm I}{}^{\rm R}\,[\mathcal{G}(u,v)]_{\rm J}{}^{\rm T}\varphi^{(2)}{}_{\rm R}{}^{\rm S}{}_{\rm T}{}^{\rm U}(III)\,\left[\mathcal{G}^{-1}(u,v)\right]_{\rm S}{}^{\rm J}\,\left[\mathcal{G}^{-1}(u,v)\right]_{\rm U}{}^{\rm L} \\
&= \varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(III) \quad , \\
\left[\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(V)\right]' &= [\mathcal{G}(u,v)]_{\rm I}{}^{\rm R}\,[\mathcal{G}(u,v)]_{\rm J}{}^{\rm T}\varphi^{(2)}{}_{\rm R}{}^{\rm S}{}_{\rm T}{}^{\rm U}(V)\,\left[\mathcal{G}^{-1}(u,v)\right]_{\rm S}{}^{\rm J}\,\left[\mathcal{G}^{-1}(u,v)\right]_{\rm U}{}^{\rm L} \\
&= \varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(V) \quad , \qquad \text{(D.4)}
\end{aligned}$$

which show that the quartic chromocharacters associated with the cases I, III, and V, possess the full SU(2) $\otimes$ SU(2) symmetry under the group element defined by (D.3). On the otherhand, we also see

$$\begin{aligned}
\left[\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(II)\right]' &= 2\,\delta_{\rm I}{}^{\rm J}\delta_{\rm K}{}^{\rm L} + \ 2\,[\,\widetilde{\beta}^1\,]_{\rm I}{}^{\rm J}\,[\,\widetilde{\beta}^1\,]_{\rm K}{}^{\rm L} \quad , \\
\left[\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(IV)\right]' &= 6\,\delta_{\rm I}{}^{\rm J}\delta_{\rm K}{}^{\rm L} \ + \ 6\,[\,\widetilde{\beta}^1\,]_{\rm I}{}^{\rm J}\,[\,\widetilde{\beta}^1\,]_{\rm K}{}^{\rm L} \\
&\quad + \ 4\,[\,\vec{\alpha}\,\widetilde{\beta}^1\,]_{\rm I}{}^{\rm J}\,\cdot\,[\,\vec{\alpha}\,\widetilde{\beta}^1\,]_{\rm K}{}^{\rm L} \ + \ 4\,[\,\vec{\alpha}\,]_{\rm I}{}^{\rm J}\,\cdot\,[\,\vec{\alpha}\,]_{\rm K}{}^{\rm L} \qquad \text{(D.5)}
\end{aligned}$$

where

$$[\widetilde{\beta}^1]_{\rm I}{}^{\rm J} \;=\; [e^{i\frac{1}{2}v^{\rm A}\beta^{\rm A}}\beta^1 e^{-i\frac{1}{2}v^{\rm A}\beta^{\rm A}}]_{\rm I}{}^{\rm J} \quad . \tag{D.6}$$

The results in (D.5) show that the quartic chromocharacters in the cases of II and IV only possess an SU(2) $\otimes$ U(1) symmetry.

We may write the transformed final chromocharacter for case-VI in the form

$$\left[\varphi^{(2)}{}_{\rm I}{}^{\rm J}{}_{\rm K}{}^{\rm L}(VI)\right]' = 3\,\delta_{\rm I}{}^{\rm J}\delta_{\rm K}{}^{\rm L} + 2\,[\vec{\alpha}\,]_{\rm I}{}^{\rm J}\cdot[\vec{\alpha}\,]_{\rm K}{}^{\rm L} + 2\,[\widetilde{\beta}^2]_{\rm I}{}^{\rm J}\,[\widetilde{\beta}^2]_{\rm K}{}^{\rm L} - 2\,[\widetilde{\alpha}^3]_{\rm I}{}^{\rm J}\,[\widetilde{\alpha}^3]_{\rm K}{}^{\rm L}\,, \tag{D.7}$$

where

$$[\widetilde{\alpha}^3]_{\rm I}{}^{\rm J} \;=\; [e^{i\frac{1}{2}u^{\rm A}\alpha^{\rm A}}\alpha^3 e^{-i\frac{1}{2}u^{\rm A}\alpha^{\rm A}}]_{\rm I}{}^{\rm J} \quad . \tag{D.8}$$

This proves that at most this chromocharacter possess a U(1) $\otimes$ U(1) symmetry.

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